Principles of Actuarial Science*

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0.1 Introduction

Actuarial science has a long history. Its foundations date back to the application of probability to insurance. The profession of actuary is over 150 years old. The actuarial profession developed its techniques mainly in the life insurance industry. These have been extended to most financial security systems including non-life insurance, social security, pensions and health care.

In recent years the quantitative approach to finance has led to the emergence of financial mathematics and financial economics as areas of study closely related to aspects of actuarial science. There are a number of differences between actuarial science and financial theory. Financial theory has studied pricing and risk management for traded financial market instruments where there has been available empirical financial market data to develop and test pricing and hedging models. Actuarial science has developed approaches to pricing and managing risks based on insurance and other financial security products where there is no actively traded market for these risks. Actuarial models have tended to address questions of risk sharing through product design including profit sharing arrangements. The need for equity of profit distribution has been a major focus. In financial theory the fair determination of prices has been a major focus since the future cash flows in financial products are usually well defined in the financial contract.

There has been a merging of concepts in finance and actuarial science over recent years. Increasingly, corporations are managing risk using enterprise wide risk management and incorporating both financial risks and insurance risks into their risk management programs. Financial intermediaries are including banking and insurance products as part of the financial services they offer. Financial market products have been developed that cover credit risks and catastrophe insurance risks. There are also specially designed products covering the total risks of a corporation available from financial intermediaries that have been established to integrate financial risk and insurance risks. These developments have led to the increased integration of actuarial science with financial economics. Finance professionals are starting to use the models developed in actuarial science to handle non-traded insurance risks and actuaries are using financial models in their work.

The actuarial profession has broadened its focus and actuaries are found in all areas of financial services. As a result, actuarial studies is now a broadly based commerce discipline with a strong quantitative foundation. Actuarial practice requires a strong foundation in economics, accounting, finance, mathematics, probability and statistics.

The topics in this book cover key concepts in probability, statistics, economics, accounting, demography, financial mathematics, actuarial mathematics and actuarial management required to understand the work of an actuary. They take a modern approach to actuarial science based on financial and stochastic modelling. Until recent years, much of actuarial science was based on the application of deterministic models assuming constant, or at least known, interest rates and valuation implicitly based on expected values, often using conservative assumptions. The developments in financial economics and the quantitative approach to financial problems over the last 40 years has provided the basis for a financial and stochastic modelling approach to actuarial science.

Developments in modern desk top computing and financial calculators has also changed the role of actuarial mathematics from that of a computational tool to that of a basis for making financial decisions involving long term contingent cash flows. Spreadsheets such as Excel and other software such as MATLAB - http://www.mathworks.com/products/matlab/, @RISK http://www.palisade.com/, O-Matrix - http://www.omatrix.com/ and Minitab allow the development of powerful financial and actuarial models to analyse problems. Computing is an essential tool for an actuarial toolkit.

Although actuarial notation is used in this book, the focus is on principles and not on memorising actuarial symbols nor on the manipulation of actuarial mathematics for its own sake. Actuarial notation was designed for the deterministic approach that the profession has now more or less assigned to its history books. Where appropriate, actuarial notation is used but understanding the concepts is more important than being able to manipulate the symbols. Students are encouraged to develop a problem solving approach.

The Society of Actuaries and the Casualty Actuarial Society is in the process of developing a document setting out the General Principles of Actuarial Science ([16]). These General Principles are the basis for a number of sections of this book. It is hoped that the book will be useful for foundation subjects in actuarial science at other Universities around the world.

Actuarial science's origins were in the study of gambling and games of chance. UNSW is located in Sydney, Australia, next door to Randwick Race Course, a horse racing venue. Thus, UNSW seems to be an ideal location for an actuarial studies program. We hope that UNSW students do not develop the habits of the early gamblers as a result!

In Australia, actuarial studies is often completed as part of a Commerce degree including subjects covering the major principles of actuarial science and subjects providing a foundation in commerce subjects such as accounting, economics and finance. Actuarial practice is usually studied through the professional body after graduating or by studying relevant post-graduate subjects at University.

This book has been primarily written for the subject ACTL1001 Actuarial Studies and Commerce at the University of New South Wales (UNSW), Sydney, Australia. It is designed to provide a foundation for the principles of actuarial science and to demonstrate the importance of the different subjects that are studied in more detail in the actuarial studies program at UNSW. It hopefully demonstrates the need for an understanding of the disciplines of accounting, economics, finance, probability and statistics in order to apply actuarial science to practical problems. Only key concepts are covered with the detail left for other subjects in probability, statistics, economics, finance and actuarial models.

The book was prepared using LaTeX. Comments on this book are welcome and should be addressed to the author at m.sherris@unsw.edu.au

Chapter 1
IN THE BEGINNING

1.1 Learning Objectives

The main objectives of this chapter are:

- to describe the early development of actuarial science from games of chance to mortality tables,
- to overview the basic principles underlying actuarial science.

This chapter provides a basic foundation for many of the topics covered in more detail in later chapters.

1.2 Early Origins

1.2.1 Probability

The mathematical foundations of actuarial science have an interesting background and can be dated back to the 1600's. They originate in gambling and the formal study of games of chance. Games of chance and other forms of gambling have always been popular with human kind. Dice games have been popular from early times. Claudius (10BC-54AD) is claimed to have even published a book on "*How to win at Dice*"! ([4]). These games involved the rolling of a number of die and the wagering of money on the outcome. Although gambling has often been illegal, in Europe in the 1500's there was a law covering "Funeral Expenses and games of chance" which allowed gambling at Funerals by law. In more recent times Casinos and other forms of gambling, horse racing and poker machines have become major revenue raisers for governments. Card games are more recent than dice games since playing cards were not invented until about 1350AD.

In order to properly analyze games of chance it is necessary to know some theory of probability. The early gamblers did not know how to formally calculate the odds of various outcomes of dice and card games, but there is some evidence that they had some idea of the relative chances based on their gambling experiences. One approach to probability is to observe the outcomes of various possible events from actual experiments and to assess the relative probability of these events based on the proportion of times the particular event of interest occurs. Thus for a dice game if the roll of a six from a fair dice was the event of interest then the probability of getting a six could be assessed by rolling the dice many times and counting the number of times a six appears. The proportion of the rolls where a six occurred would be an estimate of the probability. Another approach is to evaluate all possible outcomes and assume that each outcome was equally likely. The probability of any event could then be determined by enumerating all of the possible cases. The proportion of the total possible outcomes where the event of interest occurs is the probability of the event. It is important to note that each possible outcome must be equally likely to occur.

The theory of probability had its origins in the work of Cardano, who's book on games of chance written around 1520 was published after his death in 1663 ([3]). He was the first to calculate probabilities of various dice and card combinations in a theoretically sound way by enumeration of cases. Even Galileo (1564-1642) had turned his thoughts to games of dice providing the basis for enumerating the cases resulting from the roll of three die.

A more formal development of probability is found in correspondence between Pierre de Fermat (1601-1665), of Fermat's Last Theorem fame, and Blaise Pascal (1623-1662) about a game of dice. Neither of these were actuaries, since the concept of an actuary did not exist until 1762. Fermat was a mathematical genius of his time. Pascal, although an able mathematician in his youth, was clearly not of the same calibre as Fermat. Pascal is often associated with the famous "triangle", but this result was in fact discovered earlier by, amongst others, his teacher Hérigone who had published a table of numbers for determining the coefficients of integer binomial powers in *Cours mathématique* (1634). At the time it was common to copy the work of others and publish it slightly altered as original work, something that is not tolerated these days, especially not in actuarial assignments.

They considered the problem of two gamblers who agree to play a game of dice until one of them wins a specified number of points (n, say). If they stop after one has won x points and the other has won y points, then how do they divide the stakes? The stakes are the total amount that the gambler will win if successful, or lose if unsuccessful. This is the *problem of points.*

Exercise 1.1 *A gambler has undertaken to throw a six with a die in eight throws. Suppose she has made three throws without success. What proportion of the stake should she keep if she is to give up her fourth throw?*

Christianus Huygens (1629-1695) developed rules for solving problems of dice games and first set out the concept of expected value. Huygens work was the basis for the theory of probability for almost half a century until the work of James Bernoulli (*Ars Conjectandi*, 1713) and Abraham de Moivre (*Doctrine of Chances*, 1718). Huygens gave as an exercise in his *Games of Chance* an example of the "gambler's ruin" problem.

James Bernoulli (1654-1705) generalised the gambler's ruin problem to the case where A has m units to stake, B has n units and the chances of A or B winning a game are in the ratio $a : b$. This appeared in *Ars Conjectandi* which was not published until after his death in 1713.

Actuarial science is founded on probability theory and its application to human financial affairs. The scientific approach to life insurance was developed by James Dodson who was a pupil of Abraham de Moivre (1667-1754). de Moivre had developed the first treatment of probability in English, the *Doctrine of Chances*, and applied the theory of probability to problems related to annuities on human lives in his *Annuities upon Lives*. de Moivre was a friend of Edmond Halley and Isaac Newton. The first edition of his *Doctrine of Chances* was dedicated to Newton. Dodson had developed a table of decrements based on the Bills of Mortality of London and used this to calculate life insurance premiums for different ages.

1.2.2 Mortality and Compound Interest

Understanding the theory of probability involved in games of chance is not enough to analyze actuarial problems. It is also necessary to allow for compound interest and to have a basis for estimating the relevant probabilities. Richard Witt published a book on compound interest in 1613 that is often regarded as an earlier contribution to actuarial science.

John Graunt (1620-1674) had been the first to produce a life table based on the Bills of Mortality of London. The Bills of Mortality included details of the deaths each week and the cause of death. They were originally designed to alert people to diseases such as the plague. Graunt analyzed the deaths and attributed them to the older and younger ages based on the cause of death. He then produced a life table showing the number alive at various ages and the number of deaths in various age groups. His life table is shown below. The starting point of 100 is referred to as the *radix* of the table.

Age	**Number alive at age**	**Deaths in age group**
0	100	36
6	64	24
16	40	15
26	25	9
36	16	6
46	10	4
56	6	3
66	3	2
76	1	1
80	0	-

Exercise 1.2 *Using John Graunt's life table, calculate*

1. *the probability that a life currently aged 16 will be alive at age 66,*

2. *the probability that a life aged 36 will die between ages 46 and 56, and*

3. *the approximate average life time of a life aged 0.*

Johann de Witt (1625-1672), who was the Prime Minister of the Netherlands, had shown how the chances of death and the principles of compound interest could be combined to value a life annuity. Annuities are streams of regular payments. A life annuity is an annuity that is paid as long as a particular life is alive. As soon as the life dies then the annuity payment ceases. Life annuities are contingent payments since the payment of the regular amount of the annuity is contingent on the life being alive. deWitt's main interest in this problem was to determine the price to use in raising money from the sale of life annuities.

Governments often used life annuities to borrow money. de Witt calculated that at 4 per cent per annum interest the government should charge a purchase price of 16 times the annual amount of the annuity for a life aged 3 years old. de Witt assumed that of a group of 768 lives aged three, six would die in each half-year for the next 50 years, then 4 each half-year for the next ten years, then three each half-year for the following ten years and finally two each half-year for the final 7 years. Lives who purchase life annuities will usually be in reasonable health since the payments received will be greater the longer you live. If the deaths used in calculating the value of the annuity are based on the mortality experienced by all lives in a city then the amount charged for the annuity is likely to be too low. This effect is referred to as *self selection* since the lives purchasing the life annuities self select to purchase the annuity if they consider their chance of survival to be good. Lives who consider their survival chances as low are less likely to purchase a life annuity.

Exercise 1.3 *Locate a mortality table of the mortality experience of the lives who have purchased annuities from an insurance company in a particular country. Also obtain a life table for the population of this same country. Make sure that the two life tables are for the mortality experience over the same time period and for the same sex. Compare the two mortality tables and comment on any selection of lives that appears to have occurred in the annuity mortality tables.*

Edmond Halley (1656-1742), of Halley's Comet fame, had developed a mortality table from the bills of mortality of the town of Breslau and used his table of mortality to calculate values of life annuities. His work was published in 1693. Halley also considered the value of annuities whose payment depended on the survival of two lives, called *joint life annuities*. Halley's work in this area was probably inspired by his friendship with de Moivre. In 1725 de Moivre published a method for calculating life annuity values where he assumed that the number alive according to the life table decreased in arithmetical progression.

1.2.3 Development of Insurance and Pensions

Life insurance appears to date from the 1500's with the earliest known life insurance policy issued in June 1583 for a term of 12 months at a premium of 8%. Life assurance was offered by the Royal Exchange and London Assurance as one-year contracts up until the 1720's. These contracts were renewable yearly and the premium increased with age. The Amicable started offering long term insurance contracts for the first time in 1706. The basis for determining premiums and payments on death was not well founded and the company ended up with financial problems.

The Deed of Settlement of the Society for Equitable Assurance on Lives and Survivorships, executed in 1762, created the position of *Actuary* as the Chief Executive of the new Society. The use of the word *actuary* is attributed to Edward Rowe Mores. The word *actuary* derives from the Latin *actuarius*, who was the recorder of the proceedings of the Senate under the Roman Empire. This use of the word was probably motivated by the fact that the main duties of the actuary would be to record the details of the contracts made by the Society. The Equitable was to issue long term life insurance contracts based on the premium table developed by James Dodson. Dodson was a mathematician who had developed a basis for offering long term whole-of-life insurance with level annual premiums. The premiums would vary by age at which the life insurance policy was purchased but would remain level throughout the life of the contract. Dodson used mortality rates based on the Bills of Mortality for the City of London for the period 1728-50.

Modern non-life insurance appears to have originated in the 1300's when wealthy individuals financed sea voyages and agreed that if the ship was lost then they would not be repaid their funding of the voyage. If the cargo arrived safely then interest paid on the loan was at a higher than normal interest rate to cover the risk of loss. In the 1500's the premium rates for marine insurance varied between 3 and 15 per cent of the value of the goods for one-way voyages [13].

The first public-service pensions were paid in the late 1600's and early 1700's. These were financed on a *pay-as-you-go* basis where the contributions made by the members of the fund were used to pay the pension benefits to the retired members. As the number of pensioners increases relative to the contributing members the *pay-go* contributions must also increase. The alternative is to build up a contribution sufficient to meet the expected costs of paying the pensions of each group of members. This is referred to as *funding* the pension.

In 1747 Corbyn Morris published *An Essay towards illustrating the Science of Insurance* [13] in which a fundamental principle of insurance was set out. Morris demonstrated that an insurer's *probability of ruin* decreases as the number of independent policies increases for the same total amount of premium. The probability of ruin is the chance that the company will not have sufficient funds from its own capital and from the premiums it charges to meet the cost of the claims.

Exercise 1.4 *Assume that claims on an insurance company during a year for a par-*

ticular insurance policy are independent events. Assume also that the chance of a claim is 0.1 and that the sum insured for each policy is \$100,000. The company charges a premium of \$15,000 for each policy and contributes capital of \$20,000 in total. If a company writes only one insurance policy, what is the chance that the company will be ruined? What is the chance of ruin if the company writes ten insurance policies? How would you determine the probability of ruin if the company writes n insurance policies?

Through the 1800's and the 1900's the actuarial profession has developed and extended the techniques that originated in the 1700's in the application of probability to insurance. The establishment of the actuarial profession occurred in different countries at different times. The Institute of Actuaries in the UK was formed in 1848 and The Institute of Actuaries of Australia dates its origins back to 1897.

During the 1800's the main focus of the profession was on the actuarial management of life insurance companies. The long term contracts included whole-of-life, endowment, pure endowment and annuity policies. The premiums charged for these contracts were based on population deaths and it was found that the mortality assumptions were too high since the life insurance companies selected the healthier lives for insurance and refused insurance to the less healthy. This process is referred to as *underwriting*. As a result, life insurance companies found that they had *surplus* funds over and above that required to pay the estimated future claims on their policies. The surplus was considered to belong, at least in part, to the policyholders who had paid too much for their insurance contracts, and a basis for returning the surplus had to be developed.

Exercise 1.5 *If using population mortality for a life insurance policy results in a surplus then what would you expect to happen with life annuities if you use the population mortality table to calculate life annuity rates.*

The equitable distribution of surplus became a major focus of the actuarial profession and the fair treatment of policyholders and shareholders was a critical role played by the actuary of the insurance company. As well as assessing the long term liabilities of the insurance company, the actuary had to ensure that the surplus arising from the experience being more favorable than allowed for in the premiums charged as distributed in a fair and equitable manner.

Since the risk in a long term life insurance contract increases as the chance of dying increases with older ages, a level premium throughout the term of the contract will be higher than the charge required to cover the risk for younger ages and lower than required to cover the risk at the older ages. This leads to a build up of the excess of the premium over the expected cost of the risk in the early years of the contract that must be invested to be available to be drawn on when the premium is insufficient to meet the expected cost of the risk. The actuary must consider the investment strategy that will best ensure that the liabilities to pay insurance benefits in the future will be met.

Because the future is not predictable, it is not possible to determine exactly how much will be required to meet the future insurance payments in respect of claims. The actuary needs to estimate the amount required to ensure that the life insurance company will be able to pay the claims with a high probability. If a life insurance company is able to pay its future claims and continue to issue new business and can continue to operate soundly, even in adverse circumstances, then it meets *capital adequacy* requirements. If the company is expected to have difficulty in meeting its future claims then it can be regarded as insolvent in the sense that its future asset cash flows are highly unlikely to be sufficient to meet its future liability cash flows unless it obtains some more capital or the future circumstances change from those expected.

From the early 1900's the actuarial profession has developed the scientific basis for non-life insurance although there were some notable earlier contributions to this area, including that mentioned above by Morris. The mathematical foundations of risk theory were developed by the European actuaries.

Throughout the 1900's the actuarial profession has taken a leading role in the development of the pensions and superannuation funds used mainly by large companies and the public service to provide retirement benefits for their staff. The last 10 years or so has seen an increasing proportion of the population making provision for their retirement through superannuation funds as the need to provide for the costs of retirement and old age is increasingly being shifted from the social security system to private pension provisions. The funding of health care for the aged is also an important area of actuarial involvement.

1.3 Basic Principles

The Society of Actuaries and the Casualty Actuarial Society have developed a document setting out the scientific framework underlying the actuary's work. A Discussion Draft ([16]) sets out the principles on which actuarial science is based. Actuarial science is an applied science. The principles are founded in mathematics, statistics, economics and finance and the applications are in those involving risk and uncertainty particularly in financing of future contingent events such as death, earthquakes, and retirement. The subsequent chapters will explore these principles in more detail. They will also introduce the application of the principles to actuarial practice.

1.3.1 Deterministic and Stochastic Models

Actuaries use these principles along with actuarial models of risk as to the occurrence, timing and severity of future events in financial security systems to understand the financial implications of these risks and to help manage these risks. Actuaries use both deterministic and stochastic models. *Stochastic* models are probability based models where outcomes can take a range of values assumed to be generated by a probability distribution or multiple probability distributions. *Deterministic* models use only one outcome to model the future. This would usually be the most likely outcome, but it could also be a best case or worst case outcome. Several different deterministic outcomes could be used in modelling. In this case the analysis is referred to as *scenario analysis*. Stochastic models are usually better for understanding the amount of variation in the future. If this variation is small, then a deterministic model may be adequate.

1.3.2 Probability and Statistics

Probability and statistics are fundamental to actuarial work. More advanced probability modeling is covered later in an actuary's training. Actuaries work with distributions of future outcomes. Future outcomes are treated as *random variables.* The random variables considered by actuaries include both financial and insurance random variables. Insurance random variables are mainly related to losses that arise from insurable events such as fire or automobile accidents. Financial random variables can include favorable outcomes as well as adverse outcomes as in the insurance loss case. Thus the future profit from the sale of an insurance contract would be a financial random variable. In fact, insurance random variables can be treated as financial random variables which are restricted to negative values. However, in insurance modelling, loss random variables are treated as positive random variables.

1.3.3 Risk Pooling

Insurance risks are *pooled* in an insurance company. Individuals and companies purchase insurance policies and the premiums paid for these policies are invested into a fund in the insurance company. Claims and expenses are paid from the fund. The policies form a pool of risks. These risks are affected by differing *risk factors* but many of the risks share common risk factors. For instance, the age and sex of the driver of a car may be risk factors affecting the losses that arise in comprehensive motor vehicle insurance. If we were to assume that the claim rate was the same for all drivers of the same age and sex and that the driver's were *independent* risks, then the pooling of risks will be beneficial since only a proportion of the drivers will be expected to claim at any one time. The proportion of drivers who make claims will be more predictable the larger the number of insured drivers in the insurance pool.

For life insurance, key risk factors are age and sex. Older lives suffer heavier mortality. Similarly, in many countries, male mortality is higher than female mortality. The issue of risk factors and their use in insurance is important. Anti discrimination legislation often does not favour discrimination in insurance and annuity prices unless there is data to support different rates for different sexes.

Exercise 1.6 *Insurance company premium rates for term insurance for a given age differ depending on the sex of the life insured. Female term insurance rates are lower. An insurance company wishes to remove discrimination in its underwriting by introducing unisex term insurance premium rates. If other companies maintain* *lower rates for females, discuss what you would expect to happen if the company introduces unisex rates based on the average of the male and female rates.*

1.3.4 Law of Large Numbers

Insurance relies on *expected values* and the statistical regularity of loss random variables sometimes referred to as the *Law of Large Numbers*. The *expected* value of a random variable is the probability weighted average of the possible values that the random variable can take. There are many forms of the Law of Large Numbers. In insurance, the law of large numbers that is required basically states that as the number of claims increases the variability in the average claim decreases. If a large enough number of events are observed then the average claim will be a good estimate of the contribution of each individual risk to the total claims. Insurance also relies on the *independence* of individual risks. If the occurrence of insurance claims are independent then the law of large numbers will produce an averaging of risk for the insurance claims. If risks are not independent then there will still be risk averaging for individual risks provided they are not *perfectly correlated*. Whenever risks are exposed to the same risk factors there will usually be some correlation of risk. In financial markets, the risk arising from economic factors are usually correlated. For this reason, investment risk usually has different risk properties to insurance risks.

In calculating the premium to charge for an insurance risk, it is not sufficient to charge the average claim cost as the premium for each policy even if the law of large numbers applies. It is necessary to charge a loading to the average claim cost as the premium if the insurance company is not to be ruined in the long run.

1.3.5 Interest rates and economics

The economics underlying actuarial science are the foundations for some very important actuarial principles. A key actuarial principle from the early days of actuarial involvement in insurance and finance has been the time value of money. In economic theory, interest rates represent the trade-off between present and future consumption. An interest rate is an exchange rate between current and future consumption. Individuals will have different preferences for future and current consumption. Some will wish to consume more now and will want to borrow against future consumption. Others will prefer to save for future consumption and sacrifice current consumption. These individuals will borrow and lend through the financial market and financial intermediaries such as superannuation funds, insurance companies and banks. In order to borrow funds they will create financial liabilities and in order to invest funds they will create financial assets. In economic theory, the interest rate for borrowing or lending represents the market equilibrium that results from the balancing of the supply and demand for financial assets. The interest rate represents the *marginal rate of substitution* between consumption now and consumption in the future. In equilibrium, the marginal rate of substitution is the same for all investors and is equal to the market rate of interest.

The theory of interest has been a fundamental area of importance for actuaries since the earliest days. Actuarial practice is concerned mainly with long term financial products and rates of interest are an important factor in evaluating such products. Interest rates are used to present value future cash flows. This involves converting future monetary amounts to values expressed in today's dollars. These cash flows occur at different times and the interest rate will usually vary with the time to receipt of the cash flow. The pattern of interest rates for varying maturities is referred to as the *term structure of interest rates.* Any adjustment for time value of money will include an allowance for expected inflation in the interest rate. Thus the interest rate can be considered as an allowance for inflation plus an additional time value component. The additional component is usually referred to as the *real rate of interest.*

1.3.6 Risk and Economics

Another important factor in determining present values of future cash flows is the level of uncertainty in the cash flow. Uncertain cash flows are modelled by assuming that the amount of the cash flow is a random variable. If we know or can assume a distribution of the future cash flow then we can evaluate the expected value of the cash flow. The rate of interest used to value different uncertain cash flows that are paid at the same future time will differ if these cash flows are affected by different risk factors. Even if the cash flows have the same expected value the interest rate may vary. This is because interest rates will need to allow for the level of risk in the future cash flow.

The economics of risk are very important to actuarial science. *Utility theory* provides a basis for quantifying risk that dates back to the earliest analysis of games of chance and gambling. Calculating the expected value of a gamble or insurance risk is not sufficient when it comes to quantifying risk. The statistical distribution of the risk, and not just the expected value, and the level of *risk aversion* of an individual will affect the value of a gamble or risk. In order to quantify risk a standard approach is to assume that individual decision-makers use an *expected utility principle.* If we assume certain axioms of behaviour towards risk then it is possible to show that individuals will use expected utility to compare different risky alternatives. Thus instead of evaluating the expected values of a risk, individuals will evaluate the expected utility of the risks and use this to compare the level of risk of alternatives with uncertain outcomes.

The *certainty equivalent* of an uncertain cash flow is the amount that an individual will accept as a certain amount in place of an uncertain cash flow. Thus the expected utility of the certainty equivalent will be equal to the expected utility of the uncertain cash flow.

1.3.7 Financial Principles

Insurance companies receive premiums and invest them in financial and other assets in order to meet future liabilities. Pension and superannuation funds also invest contributions in financial markets. The balance sheet of these financial intermediaries consists of assets and liabilities. The liabilities are the future obligations to pay benefits under insurance policies and trust deeds of superannuation funds or other contractual liabilities. The valuation of these liabilities is an important role of the actuary to these funds.

The assets are mostly invested in financial instruments. Financial instruments usually trade in financial markets. These financial markets are often electronic markets or markets where trading takes place over the telephone. The trading floors of the stock exchanges and other trading exchanges have been increasingly replaced with computer based trading. Regardless of where these financial instruments trade, the financial markets provide valuable information about the market prices of financial assets. This information is readily available from newspapers, central bank and stock exchange publications as well as through the WWW. The market values represent the price that they can be bought or sold in a particular market. Some of the assets of these funds are also held in *real assets* such as property which are not traded on financial markets.

Accounting for the assets and liabilities of financial intermediaries has moved increasingly from a *book value* or *historical cost accounting* basis to *market value based accounting*. The balance sheet for insurance companies and pension funds would usually record the value of financial assets at the cost when purchased. Some of these assets would be written up or down to their maturity value but usually the difference between the market value of the assets at any time and the cost was a hidden reserve. Since most assets would appreciate in market value this reserve could be quite large, especially if the fund had a *buy and hold* investment strategy. There has been an increased use of market values to account for financial assets and an increased awareness of the need to value the balance sheet items of financial intermediaries at market value. This raises interesting issues since the liabilities of these financial intermediaries do not trade in a market.

In order to value these liabilities it has been necessary to develop financial models that can produce *fair values of liabilities* consistent with the market values of the assets. It is necessary to understand *asset pricing models* used in financial economics in order to develop such fair value models. These asset pricing models are based on both *arbitrage-free models* and *equilibrium models*. They assume that market prices satisfy economic assumptions. For example, that it is not possible to trade at the market prices and to generate a profit without taking any risk (arbitrage-free models) or that the net supply or demand for financial assets at the market price is zero (equilibrium models). Financial economics is fundamental to the valuation of liabilities and to an understanding of the market valuation of traded financial assets.

1.3.8 Actuarial Modelling Principles

The most fundamental of actuarial models consists of a probability model of actuarial risk variables and an economic or financial model of economic variables. These components of the actuarial model are the actuarial assumptions underlying the model.

Actuarial models are used to quantify future uncertainty and to present value *contingent future cash flows*, especially those depending on actuarial assumptions. Contingent cash flows are those dependent on future events. In life insurance the contingent events that the insurance cash flow depends on are the death or survival of a single life or multiple lives. In a pension fund the contingent events include the resignation of the member of the pension plan or the early retirement due to ill-health. Cash flows in insurance products can also be contingent on financial and economic variables. For example, annuity payments are often indexed to an inflation index but with a minimum guaranteed rate of increase. The annuity cash flows are contingent on future inflation in this case.

Actuarial modelling is both an art and a science. Practical experience provides the judgement skills required to implement models in practice. Theory is important but in order to implement the theory it is necessary to take into account various factors not formally included in an actuarial model. Professional codes of conduct and standards of the actuarial professional body set out the practice requirements for actuaries. It is also necessary to be aware of business ethics and to behave accordingly.

1.3.9 Financial Security Systems and Risk Management

A *financial security system* is a system where benefits are provided to members of the system in return for financial consideration. The financial consideration could be payable as a single payment or spread across time. Benefits are usually contingent on various risk factors occurring such as death, sickness or age retirement. Membership of financial security schemes can be voluntary or mandatory. Financial security systems include insurance, social security and health care systems.

Risk management involves risk identification, risk assessment, risk control or risk financing for future uncertain and contingent cash flows. It is necessary to identify the risk factors influencing a particular risk and to assess or quantify the risks. If there is available sufficient historical data then statistical analysis can be used to assess the risk. In other cases it is necessary to design a risk control and risk financing mechanism in a financial security system that can manage the future uncertainty. Actuaries have used various *profit sharing* arrangements for this purpose. These include the bonus schemes of traditional life insurance policies.

A key risk management tool is the *pooling of risk*. This is a fundamental principle underlying insurance where the pooling of independent risks results in an averaging of risk. The larger the number of similar but independent risks in a risk pool, the lower the variability of the average risk. *Diversification of risk* arises when less than perfectly correlated risks are pooled. The extent of the diversification benefit depends on the degree of correlation between the risk factors.

An important factor in financial security systems is the *solvency* of the system. The system will only deliver the benefits that are promised if there are sufficient funds available to pay these benefits in the future. Since the amount of these benefits is uncertain and contingent on future outcomes it is necessary to assess the likelihood that the benefits will be paid in adverse circumstances and, if possible, ensure that the risk control and risk financing mechanisms will ensure the solvency of the system. One method of ensuring this is to charge safety loadings or safety margins in the premiums for financial security products. As the experience unfolds in the future it may be possible to return these safety margins as profits if the adverse conditions that could lead to the insolvency of the financial security system do not eventuate. This has been the basis of many actuarial risk control and financing systems.

Risk classification systems aim to separate individual risks into groups with similar risk characteristics. These can then be treated as a group for the purposes of statistical assessment of experience and for determining the consideration to be paid for the financial security benefits. For instance, in life insurance a rating method is often used to assess lives based on blood pressure, weight, height and other health and life-style factors such as smoker status. Once the rate is set for this group it is also possible to adjust the rate according to the experience of an individual or group of individuals. These adjustments are referred to as *experience rating*.

Any financial security system must deal with the issues of *moral hazard* and *adverse selection*. Moral hazard occurs where payments are made due to the careless or deliberate actions of the members of the financial security system. These actions usually only occur because of the existence of the financial security system. Thus in fire insurance, an insured may not take precautions against fire because of the existence of the insurance or may even cause a fire in order to claim on the insurance. Adverse selection is where worse than average risks have incentives to be members of the financial security scheme. If a premium rate for insurance is based on average mortality rates then those with lower than average mortality will tend to purchase less insurance than those with higher than average mortality. Adverse selection relies on individuals having more information about their risk than those offering the financial security products. This is referred to as *asymmetric information*.

1.4 Computing Tools

There has been a rapid development in the power of desktop computers over recent years. The range of software available to model and analyze financial and actuarial problems has exploded. Time spent mastering a spreadsheet such as Excel, a statistical package such as Minitab or SAS, and a numerical computing package such as MATLAB (http://www.mathworks.com/products/matlab/) or O-Matrix (http://www.omatrix.com/) will be well rewarded. These packages also provide a useful environment to experiment with new concepts and to assist in understanding and learning. There are many books covering the use of these packages that are easy to work through and provide a lot of useful tricks and techniques for getting the most of these packages that you would not learn from a formal course. For Excel, the book by Walkenbach[18] is very good.

1.5 Conclusion

This chapter has provided an overview of the early origins of actuarial science. It has also summarized the key principles underlying actuarial science and its application to problems in the risk management of financial security systems.

1.6 Further reading

Ogborn ([11]) provides a coverage of the early development of the first insurance company and the early actuaries. David ([4]) is an excellent account of the early development of probability including translations of the original correspondence between Pascal and Fermat. The SoA/CAS Discussion Draft on General Principles of Actuarial Science is worth reading for a more detailed coverage of the basic principles.

1.7 Solutions to Exercises

Ex 1.1 *Consider the following: For a single throw the odds of throwing a six are* $1:5$ *since there is a* $\frac{1}{6}$ *chance of throwing a six. If the stake is* S *and the gambler gives up the throw then the stake should be divided in the ratio* $1:5$. *The gambler will win if she throws at least one six in eight throws. The probability of no sixes in eight throws is* $\left(\frac{5}{6}\right)^8 =: \frac{390\,625}{1679\,616}$ *so that the probability she will be successful in eight throws is* $1-\left(\frac{5}{6}\right)^8 =: \frac{1288\,991}{1679\,616} =: .767\,43$.*)*

Ex 1.2 *The probability that a life currently aged 16 will be alive at age 66 can be calculated by noting that in John Graunt's life table out of 40 aged 16 there are 3 still alive at age 66. Therefore the required probability is*

$$\frac{3}{40} = 0.075$$

The probability that a life aged 36 will die between the ages of 46 and 56 is determined by noting that out of 16 lives alive at age 36, 4 die between the ages of 46 and 56 so that the required probability is

$$\frac{4}{16} = 0.25$$

The approximate average life time of a life aged 0 can be worked out by assuming that the deaths in any period occur at the midpoint of the age interval. Thus we would assume $\frac{36}{100}$ *have a life of 3 years,* $\frac{24}{100}$ *have a life of 11 years,* $\frac{15}{100}$ *have a life of 21 years,* $\frac{9}{100}$ *have a life of 31 years,* $\frac{6}{100}$ *have a life of 41 years,* $\frac{4}{100}$ *have a life of 51 years ,* $\frac{3}{100}$ *have a life of 61 years,* $\frac{2}{100}$ *have a life of 71 years and* $\frac{1}{100}$ *has a life of 78 years. To calculate an approximate average we just weight the life times in years by* *the proportion living that long so that*

$$\begin{aligned} \textit{Approximate average life} &= \frac{36}{100}3 + \frac{24}{100}11 + \frac{15}{100}21 + \frac{9}{100}31 \\ &\quad + \frac{6}{100}41 + \frac{4}{100}51 + \frac{3}{100}61 + \frac{2}{100}71 + \frac{1}{100}78 \\ &= 18.2 \textit{ years} \end{aligned}$$

Ex 1.3 *Hint: the Society of Actuaries WWW site http://www.soa.org has a Table Manager with many different mortality tables available.*

Ex 1.4 *The company with one insurance policy will have premium of \$15,000 and capital of \$20,000, a total of \$35,000. With one policy there is a chance of 0.1 that the company will pay out \$100,000 (a claim occurs) and a chance of 0.9 that the company will pay out zero (no claim occurs). If the company pays out a claim then it will be ruined since \$100,000>\$35,000. So probability of ruin is 0.1. If the company writes 10 insurance policies then the premium income will be \$150,000 and the capital of \$20,000 will mean that the company has a total of \$170,000 in cash to meet its claims. The chance that there are j claims from these 10 policies (with a total claim payment of* $j100,000$*) is worked out as follows:*
If j policies have claims then 10-j do not have claims and the chance this occurs is $0.1^j 0.9^{10-j}$ *(which assumes independence of claims).*
The j policies with claims can be selected from the 10 policies in $\binom{10}{j}$ *ways (10 choose j).*
Thus the probability of j claims from these 10 policies is $\binom{10}{j} 0.1^j 0.9^{10-j}$.
The company will be ruined if there are now 2 claims or more which is 1 minus the probability of none or 1 claim. The probability of zero claims is $0.9^{10} = 0.3487$ *and the probability of 1 claim is* $10 \cdot 0.1 \cdot 0.9^9 = 0.3874$ *so that the probability of 2 or more claims is* $1 - 0.3487 - 0.3874 = 0.2639$.
If the company writes n insurance policies then we just work out the premium income from n policies as $n15,000$ *and add the capital of \$20,000. We then work out the number of claims required to ruin the company as*

$$i = \frac{n15,000 + 20,000}{100,000}$$

The probability that there are i or more claims will be the probability of ruin. This will be equal to

$$\sum_{j=i}^{n} \binom{n}{j} 0.1^j 0.9^{n-j}$$

We would need to use a computer program or a spreadsheet to determine this. (As an exercise students may try writing a visual basic program or setting up a computer spreadsheet to calculate these probabilities for various values of n).

Ex 1.5 *In this case we have an insurance market where one company will offer unisex rates which will be an average of the male and female rates. They will be higher than the female only rates and lower than the male only rates. The other companies will offer lower rates for females and higher rates for males. In these circumstances very few females will purchase insurance from the company with the unisex rates since they can buy cheaper elsewhere. But lots of males will be attracted to the lower rates of the unisex company. Since the company is only charging for an average risk and most of the policies will be on male lives with higher risk than is being charged, it will not be long before the company goes broke or withdraws from the insurance market.*

Chapter 2
GAMES OF CHANCE

2.1 Learning Objectives

The main objectives of this chapter are:

- to introduce some basic principles of probability, and
- to indicate how probability is applied to actuarial problems.

2.2 Probability

The early development of actuarial science originated from the analysis of dice games. An understanding of how probability is applied to games of chance assists in understanding how probability is applied to actuarial problems. Games of chance and insurance share common probabilistic foundations. However games of chance are usually more fun to analyze. In actuarial science the probability of interest is that of various events related to the death or survival of one or more human lives or events such as car accidents or storms. In games of chance we are interested in the probability of various outcomes such as the total of the numbers appearing on the face of several dice.

2.2.1 Permutations and Combinations

To begin with it is necessary to have a basic understanding of permutations and combinations. We are often interested in the number of ways that different objects can be arranged. These different arrangements are called *permutations*.

If we have n distinct objects then we can arrange these in $n!$ (n factorial) permutations. This is simply because there are n objects that we can select as the first object, $n-1$ objects we can select as the second object and so on until the last object. Thus the number of permutations is $n.(n-1).(n-2)\ldots 2.1$ which is n factorial. If we arrange the n distinct objects in a circle then the number of permutations will be $(n-1)!$. If there are r objects which are the same then the number of permutations is $\frac{n!}{r!}$.

Example 2.1 *How many distinct ways can you arrange the letters of the word ACTUARY?*

Solution 2.1 *There are 7 letters but there are 2 letter A's. The answer is* $\frac{7!}{2!} = \frac{7.6.5.4.3.2.1}{2.1} = 2520$

Exercise 2.1 *How many distinct ways can you arrange 10 different coloured beads in a necklace?*

If we are interested in selecting r objects from n distinct objects without regard to the order in which they are selected then these are called *combinations*. The number of combinations of n distinct objects taken r at a time is $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Example 2.2 *How many different ways can 13 cards be selected from a pack of 52 cards?*

Solution 2.2 *Answer is* $\frac{52!}{13!39!} =: 635\,013\,559\,600$

Exercise 2.2 *How many ways can you select 4 subjects out of 10 subjects so as to always include a specific subject?*

Exercise 2.3 *How many ways can 20 people be seated at 2 round tables with 10 on each table?*

A useful result is the Binomial Theorem which states that

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

Example 2.3 *Simplify* $\sum\limits_{r=0}^{n} \binom{n}{r}$.

Solution 2.3 $\sum\limits_{r=0}^{n} \binom{n}{r} = \sum\limits_{r=0}^{n} \binom{n}{r} 1^{n-r} 1^r = (1+1)^n = 2^n$.

Example 2.4 *Evaluate* $\sum\limits_{r=0}^{n} \binom{n}{r} p^{n-r} (1-p)^r \quad 0 < p < 1$

Solution 2.4 $\sum\limits_{r=0}^{n} \binom{n}{r} p^{n-r} (1-p)^r = (p+1-p)^n = 1^n = 1$.

Exercise 2.4 *Show that* $\sum\limits_{r=0}^{n} r \binom{n}{r} (1-p)^{n-r} (p)^r = np$

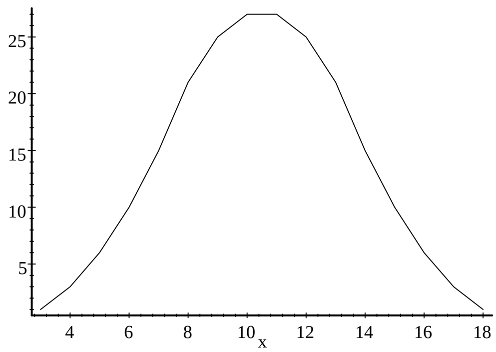
2.2.2 Probability

The probability of a particular event can be evaluated in a number of ways. One is to evaluate all possible cases. Assuming that each case is equally likely, it is then possible to determine the probability of a particular event or set of events as the proportion of total outcomes that these particular events represent. Another approach is to run an experiment and observe the frequency or proportion of times that the event of interest occurs. It is also possible to form subjective probabilities of certain events. This amounts to the use of intuition rather than empirical data. Formally in probability we need to specify a sample space of all possible events such that the probability of any event is positive, all of the events are mutually exclusive, and the sum of the probabilities is one.

Example 2.5 *What is the probability of rolling a total of 12 with three dice?*

Solution 2.5 *We can enumerate all of the possible combinations from rolling three dice. The first dice can have any of the numbers 1 to 6 as can each of the other 2 dice. Thus the total number of possible outcomes is* $6.6.6 = 216$. *We enumerate all of these and add up the total of the faces on the three die for each case to determine how many sum to 12. This could be done manually as it would have had to have been done in the early days or more easily in a spreadsheet. The results are given in the following histogram along with a plot. Note how the plot of the outcomes has a "bell" shape. We can now calculate the probability of a sum of 12 occurring as* $\frac{25}{216}$ *assuming each possible outcome is equally likely.*

Sum	*Frequency*
3	1
4	3
5	6
6	10
7	15
8	21
9	25
10	27
11	27
12	25
13	21
14	15
15	10
16	6
17	3
18	1
Total	216



The probability could be estimated by taking 3 die and rolling them 100 times and noting the number of times that a sum of 12 was obtained. The ratio of the number of times this occurred to 100 would be an estimate of the probability of a sum of 12. If we were to attempt this experiment several times more, each time with another 100 rolls, then the estimates would most likely differ. This is called sample random variation since each of our rolls of 100 produces a sample of possible outcomes from which we calculate an estimate of the proportion of times the die sum to 12. If we were to do the experiment with 1,000,000 rolls and repeat this many times then our estimate of the proportion of times the sum of the die is 12 will vary much less than compared with when there were only 100 rolls. This effect is referred to as the Law of Large Numbers. Since we are repeating the experiment many more times and taking a ratio then our estimate should vary much less from one sample to another for the 1,000,000 rolls.

We can also note that the number of ways of obtaining a total of s from t throws of a die is the coefficient of x^s in the expression

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^t$$

For $t = 3$ we could expand this expression and we will find the coefficients should agree with those given in the solution. In particular for $s = 12$ we will obtain a coefficient of 25.

Exercise 2.5 *Expand $(x + x^2 + x^3 + x^4 + x^5 + x^6)^3$ and confirm the results given in the solution above.*

Exercise 2.6 *Three dice are thrown. What is the probability that the sum of their faces is between 7 and 13?*

Exercise 2.7 *Three coins are tossed. What is the probability that they are all heads or all tails?*

Exercise 2.8 *Cards are drawn from a deck of 52 playing cards with replacement. What is the probability that you will get at least one ace in n draws? How many draws are required for this probability to exceed $\frac{1}{2}$?*

2.2.3 Random variables

A random variable X is a function that assigns values to possible outcomes. Random variables can be either *discrete* or *continuous*. Discrete random variables take a fixed number of finite values or have countably infinite possible values. Continuous random variables take values on a continuous scale. Consider the rolling of three die. The possible outcomes are the different combinations of the numbers showing on the face of the three die. The sum of the faces of the three die is a discrete random variable.

Example 2.6 *Give examples of discrete random variables in actuarial science.*

Solution 2.6 *Examples would be the number of motor vehicle accidents occurring during a particular month and the number of lives who die from cancer who are aged 40 last birthday during a particular time period.*

Example 2.7 *Give examples of continuous random variables in actuarial science.*

Solution 2.7 *Examples of continuous random variables are the time until the first accident for a particular car insurance policy or the claim payment for a fire insurance policy (although in this latter case the use of monetary units no smaller than cents means that it is strictly a discrete random variable which in practice is treated as a continuous random variable).*

2.2.4 Probability density functions

The probability that a discrete random variable X takes a particular value x is denoted

$$f(x) = P(X = x)$$

This is referred to as the *probability distribution* of X. The probability distribution has the properties

$$f(x) \geq 0 \text{ and} \sum_{all\ x} f(x) = 1$$

The *distribution function* or the *cumulative distribution* of X, denoted by $F(x)$, is the probability that the random variable takes a value less than or equal to a specified value x i.e.

$$F(x) = \Pr(X \leq x)$$

For continuous random variables the *probability density function (p.d.f.)* $f(x)$ is defined such that

$$\Pr(a \leq X \leq b) = \int_a^b f(x)dx$$

with $f(x) \geq 0$ for $-\infty < x < \infty$ and $\int_{-\infty}^{\infty} f(t)dt = 1$. The distribution function in the continuous case is

$$F(x) = \Pr(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

2.2.5 Expectation

We are usually interested in certain properties of a random variable or a function of a random variable. The *expectation* of a function a random variable $g(X)$ is defined as

$$E\left[g\left(X\right)\right] = \sum_{all\ x} g\left(x\right) f(x)$$

for a discrete random variable, and as

$$E\left[g\left(X\right)\right] = \int_{-\infty}^{\infty} g\left(x\right) f(x)dx$$

for a continuous random variable.
The *moments* (about the origin) of a random variable are defined as

$$E\left[X^r\right] \qquad r = 0, 1, 2, \ldots$$

The first moment, with $r = 1$, is referred to as the *mean of the distribution of the random variable* and is usually denoted by μ. Thus

$$E\left[X\right] = \mu = \sum_{all\ x} xf(x)$$

when X is discrete and

$$\mu = \int_{-\infty}^{\infty} xf(x)dx$$

when X is continuous. The mean is a measure of central tendency for a distribution.

The second moment about the mean is referred to as the *variance* of the distribution and is usually denoted by σ^2 Thus

$$Var\left[X\right] = \sigma^2 = E\left[(X - \mu)^2\right]$$

The *standard deviation* is the square root of the variance, σ. The variance is a *measure of dispersion* or spread of the distribution.

2.3 Some probability densities often used in actuarial science

In actuarial science there are a number of density functions applied frequently in practice. We will briefly mention a few of the more basic densities. They are:

2.3.1 Discrete

Poisson (λ)

$$\Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x = 0, 1, 2 \ldots$$

$$E[X] = \lambda$$

$$Var[X] = \lambda$$

The Poisson distribution is often used to model the probability of occurrence of rare events such as insurance claims during a particular time period. It can also be used as an approximation for the chance of dying although in theory it allows for the probability of dying more than once although the probability of this occurring should be small for a small enough parameter λ.

Example 2.8 *Assume that the probability of an insurance claim on a particular insurance policy during a year has a Poisson distribution with* $\lambda = \frac{1}{100}$. *Calculate the probability that there will be*

- *no claims during the year on the policy*
- *exactly 2 claims, and*
- *at least one claim.*

Solution 2.8 *Required probabilities are*

- $\Pr(X = 0) = \frac{e^{-\frac{1}{100}} \frac{1}{100}^0}{0!} = e^{-\frac{1}{100}} = .990\,05$
- $\Pr(X = 2) = \frac{e^{-\frac{1}{100}} \frac{1}{100}^2}{2!} = : 4.\,950\,2 \times 10^{-5}$
- $Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - e^{-\frac{1}{100}} = : 9.\,950\,2 \times 10^{-3}$

Exercise 2.9 *Show that the mean and variance of a Poisson*(λ) *random variable are equal to* λ.

Exercise 2.10 *Calculate the probability that there will be 2 or more claims if the probability of a claim has a Poisson distribution with* $\lambda = \frac{1}{50}$

Binomial (n, p)

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, 2 \ldots$$

$$E[X] = np$$

$$Var[X] = np(1-p)$$

The Binomial distribution is used for the number of successes (x) in a series of n *independent* trials where each success has probability p. In actuarial science it can be used to determine the probability that a certain number of lives in a group of lives with the same chance of dying will die over a specified period assuming independence between the lives.

Trials or events are independent if the occurrence of one does not the affect the probability of occurrence of any of the others. The probability that a number of independent events occur is just the product of the probabilities that each event occurs.

Example 2.9 *Assume that the probability of an individual aged 20 of dying in a year is* $\frac{1}{1000}$. *Calculate the probability that at least one individual will die during the year out of a group of 150.*

Solution 2.9 *The required probability is* $1 - \binom{150}{0} \frac{1}{1000}^0 \left(1 - \frac{1}{1000}\right)^{(150)} = 0.1394$

Exercise 2.11 *Show that the variance of a Binomial*(n, p) *random variable is* $np(1-p)$.

Exercise 2.12 *What is the probability that exactly 3 ones appear when 6 dice are thrown?*

Geometric (p)

$$\Pr(X = x) = (1-p)^{x-1} p \quad x = 1, 2 \ldots$$

$$E[X] = \frac{1}{p}$$

$$Var[X] = \frac{1}{p}\left(\frac{1}{p} - 1\right)$$

The Geometric distribution gives the probability of the number of trials to the first success where the probability of success on any trial is p.

Example 2.10 *Passing or gaining exemption from actuarial professional examinations is notoriously difficult. Assume that the probability of passing or gaining an exemption from a particular actuarial professional examination is* 0.6*.*

- *Calculate the probability that it will take at least 2 attempts to pass or gain an exemption from this examination.*
- *Calculate the expected number of attempts before passing.*

Solution 2.10
- *Required probability is* $1 - \Pr(X = 1) = 1 - 0.6 = 0.4$.
- *Expected number of attempts* $\frac{1}{0.6} =: 1.6667$

Exercise 2.13 *If the probability of passing an actuarial professional examination is* 0.3*, what is the probability that a student will pass an examination on their 4th attempt?*

2.3.2 Continuous

Exponential (θ)

$$\begin{aligned} f(x) &= \frac{e^{-\frac{x}{\theta}}}{\theta} \quad x \geq 0 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt \quad x \geq 0 \\ &= 1 - e^{-\frac{x}{\theta}} \quad x \geq 0 \end{aligned}$$

$$E[X] = \theta$$

$$Var[X] = \theta^2$$

The exponential distribution is used for the probability distribution of the time until an event occurs where the probability does not depend on elapsed time. It can be used to model the time of survival but not for human lives since the chance of surviving for a human will depend on the number of years since being born. We will cover some simple survival models later.

Example 2.11 *Assume that the chance of surviving for an individual does not depend on time lived and that the distribution of the survival time is exponential with* $\theta = 50$*. Calculate the probability that the individual will live more than 10 years.*

Solution 2.11 *We require* $1 - F(10) = 1 - \left(1 - e^{-\frac{10}{50}}\right) =: .81873$

Exercise 2.14 *The time (in years) until failure of a lightglobe has an exponential distribution with* $\theta = 3$*. You have just installed 10 lightglobes in your house. What is the probability that no lightglobes will last 6 years? What is the probability that exactly 4 lightglobes will last 6 years?*

$Normal(\mu, \sigma)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \text{ for } -\infty < x < \infty$$

$$E[X] = \mu$$

$$Var[X] = \sigma^2$$

The normal distribution is the "bell" shaped symmetric distribution that typically results from the accumulation of a large number of independent random events.

Note that when $\mu = 0$ and $\sigma = 1$ the density is called the *standard normal density.* Tables of the Cumulative Normal Distribution for the standard normal density $\left(N[X] = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x)^2}\right)$ are tabulated and available in most statistical texts. The cumulative density is also provided in Spreadsheets such as Excel and statistical software such as Minitab.

Note that if X is a standard normal random variable then $Y = \mu + \sigma X$ is a normal random with mean μ and standard deviation σ.

For large values of n a *Binomial* (n, p) distribution is well approximated by a $Normal(np, \sqrt{np(1-p)})$ distribution.

Exercise 2.15 *Use a Spreadsheet such as Excel or a statistical package such as Minitab to evaluate the cumulative distribution for a Binomial* $(20, 0.5)$ *random variable and for a* $Normal(10, \sqrt{5})$ *variable. Plot* $F[X]$ *and* $1 - F[X]$ *in both cases and comment.*

$Log - normal\,(\mu, \sigma)$

The normal distribution is a symmetric distribution that can take negative values. In actuarial science we are often interested in modelling such things as survival time and insurance claim amounts. These random variables are always positive and the distribution is not symmetric. In fact these distributions are positively *skewed* to the higher positive values. They are usually more peaked with a narrower "hump" and higher probabilities of extreme values than for the normal. This is referred to as lepto-kurtosis or "*fat-tails*".

If we take the exponential of a $Normal(\mu, \sigma)$ random variable then this new random variable has a $Log - normal\,(\mu, \sigma)$ distribution. The $Log - normal\,(\mu, \sigma)$ is only defined for positive values and has density function

$$f(x) = \frac{1}{\sigma x\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2} \text{ for } 0 < x < \infty$$

and

$$E[X] = e^{\mu + \frac{1}{2}\sigma^2}$$

$$Var[X] = e^{2\mu + \sigma^2}\left[e^{\sigma^2} - 1\right]$$

$$E\left[X^k\right] = e^{k\mu + \frac{1}{2}k^2\sigma^2}$$

Exercise 2.16 *The interest rate paid on a savings account each year changes through time. Assume that the annual interest rate is a random variable, R, and that $(1 + R)$ has a log-normal$\left(\ln 1.05, \sqrt{0.0004}\right)$ distribution Calculate the probability that the interest rate R will be less than* 0.04.

$Weibull\,(\alpha, \beta)$

A distribution used for survival and loss models is the Weibull. It has probability density and distribution functions as follows (here we use the same parameter specification as found in Excel and Minitab)

$$f(x) = \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha} \text{ for } 0 < x < \infty \quad \alpha > 0 \text{ and } \beta > 0$$

$$F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}$$

There are a wide range of other probability densities used for modelling insurance risks not covered here and left for later study. These include the Gamma, logGamma and Beta distributions. This has been a brief coverage of some probability distributions used in actuarial science. More detailed coverage of these is left for more advanced subjects in statistics and probability. Students interested in more details are referred to a text such as Rice [14].

We will mainly use the Binomial, Poisson, Normal and Log-normal distributions.

2.4 Gambler's Ruin

In actuarial science we are interested in both the probability of events and their financial consequences. Thus it is necessary to consider the monetary outcomes of the various possibilities. Often it is the monetary outcome that motivates gambling so we will consider some well known gambling problems that originated from the early dice games. A major consideration in a financial security system is ensuring that promises to pay benefits can be reasonably met. The chance that benefits payable from an insurance company will not be met is usually referred to as the *probability of ruin*. When a company is ruined then it has no assets available to pay its claims. Thus it is insolvent with liabilities exceeding assets.

In order to consider the financial ruin of an insurance company or other financial security system we will begin by considering the following game of chance.

> A and B start with a and b dollars respectively. They continue to play a series of games such that the probability of A winning a game is p. The loser of the game pays \$1 to the winner. The game continues until one or the other has all the money $\$(a+b)$. What is the probability that A will end up with all the money. In this case we can say that B is ruined.

In order to solve this problem we develop a recurrence relation for the probabilities. Such recurrence relations are often used in actuarial problems. Denote the probability that A will win assuming she currently has $(n+1)$ dollars by u_{n+1}.

If A wins the next throw - with probability p - then she will have $(n+2)$ dollars and a probability of winning of u_{n+2} or if B wins - with probability $q=(1-p)$ - then she will have n dollars and a probability of winning of u_n. We can therefore write

$$u_{n+1} = pu_{n+2} + qu_n \quad \text{for } 0 < n+1 < a+b$$

We can write this as

$$pu_{n+1} + qu_{n+1} = pu_{n+2} + qu_n$$

so that

$$p\left(u_{n+2} - u_{n+1}\right) = q\left(u_{n+1} - u_n\right)$$

or

$$\left(u_{n+2} - u_{n+1}\right) = \frac{q}{p}\left(u_{n+1} - u_n\right)$$

Note that if $n=0$ then A has lost and $u_0 = 0$. Working recursively we can write

$$\begin{aligned}
(u_2 - u_1) &= \frac{q}{p}(u_1) \\
(u_3 - u_2) &= \frac{q}{p}(u_2 - u_1) = \left(\frac{q}{p}\right)^2 (u_1) \\
(u_4 - u_3) &= \frac{q}{p}(u_3 - u_2) = \left(\frac{q}{p}\right)^3 (u_1) \\
&\vdots \\
(u_i - u_{i-1}) &= \frac{q}{p}(u_{i-1} - u_{i-2}) = \left(\frac{q}{p}\right)^{i-1} (u_1)
\end{aligned}$$

Summing all of these terms we get

$$\begin{aligned} (u_i - u_1) &= \left\{ \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots \left(\frac{q}{p}\right)^{i-1} \right\} u_1 \\ u_i &= \left\{ 1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots \left(\frac{q}{p}\right)^{i-1} \right\} u_1 \end{aligned}$$

Now if $p = q = \frac{1}{2}$ we get

$$u_i = i u_1$$

Note that $u_{a+b} = 1$ so that

$$u_1 = \frac{1}{a+b}$$

We require u_a which is

$$u_a = \frac{a}{a+b} \quad \text{if } p = q = \frac{1}{2}$$

Now assume $p \neq q$. We have

$$u_i = \left\{ \frac{1 - \left(\frac{q}{p}\right)^i}{1 - \frac{q}{p}} \right\} u_1$$

Using $u_{a+b} = 1$ we get

$$u_1 = \left\{ \frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^{a+b}} \right\}$$

so that

$$u_i = \left\{ \frac{1 - \left(\frac{q}{p}\right)^i}{1 - \left(\frac{q}{p}\right)^{a+b}} \right\}$$

and

$$u_a = \left\{ \frac{1 - \left(\frac{q}{p}\right)^a}{1 - \left(\frac{q}{p}\right)^{a+b}} \right\} \text{ for } p \neq q$$

Example 2.12 *Assume that you are playing a game at the casino and that the casino has wealth of \$50 million. You have wealth of \$1000 and the chance of the casino winning each game is slightly in their favour at $p = 0.51$ and you win or lose a dollar on every game. Calculate the odds that you will go broke playing at the casino (i.e. the casino will win all of your money).*

Solution 2.12 *Required probability is*

$$u_{50,000,000} = \left\{ \frac{1 - \left(\frac{.49}{.51}\right)^{50000000}}{1 - \left(\frac{.49}{.51}\right)^{50000000+1000}} \right\} =: 1.0$$

Example 2.13 *Assume in the last example that the odds were in your favour because of your card counting skills so that $p = 0.49$. Calculate the probability that you will lose all of your money.*

Solution 2.13 *Required probability is*

$$u_{50,000,000} = \left\{ \frac{1 - \left(\frac{.51}{.49}\right)^{50000000}}{1 - \left(\frac{.51}{.49}\right)^{50000000+1000}} \right\} =: \left\{4.2257 \times 10^{-18}\right\}$$

Exercise 2.17 *A and B start with 12 dollars each and continue to throw a die on the condition that if a 5 is thrown then A gives a dollar to B and if any other number is thrown then B gives a dollar to A. The game continues until one or the other has all the money. What is the probability that B will end up with all the money? In this case we can say that A is ruined.*

2.5 Simulation

One approach to evaluating games of chance and other outcomes of random events is the frequency approach. This entails carrying out an actual experiment and recording the outcomes. In the case of coin-tossing or dice-rolling it is relatively easy to carry out the experiment. Obviously it is usually better to derive the required result using a theoretical basis. The experiment will be subject to experimental error and must be carried out many time to ensure an accurate answer. In many cases it is not possible to carry out the experiment and it may not be possible to derive a theoretical result. In these cases we can simulate the process of interest without actually carrying out the experiment. This can be carried out efficiently using a computer.

Desktop computers and various software allow the evaluation of problems in risk, insurance and finance using relatively sophisticated probability models. These models can be used to simulate the events of interest. Spreadsheets and statistical software packages can simulate a range of probability distributions. There are also specially designed software packages that can be used for probability modelling using simulation (for example MATLAB - http://www.mathworks.com/products/matlab/, @RISK http://www.palisade.com/, and O-Matrix - http://www.omatrix.com/).

In order to carry out simulation we need a *random number generator*. Excel includes random number generators for the Uniform, Normal and Poisson distributions (amongst others). The uniform random number generator produces numbers that lie between 0 and 1 and with each number equally likely.

Example 2.14 *Simulate the roll of a die and calculate the expected value on the die face using 100 and 1000 rolls of the die.*

Solution 2.14 *We generate 100 and 1000 uniform random numbers. If the random number is between* $\frac{i-1}{6}$ *and* $\frac{i}{6}$ *then we assign* i *as the outcome for* $i = 1$ *to* 6. *To do this we take outcome = 1+ integer part of* $[6 * \text{rand}()]$. *To calculate the expected value we add up the outcomes and divide by the number of simulations. The average should be around the theoretical value of* $\frac{1}{6}\frac{6.7}{2} = 3.5$. *If the simulation is carried out many times then you should notice that the estimate based on 100 simulations shows much greater variation than that based on 1000 simulations.*

Exercise 2.18 *Assume that the number of insurance claims in a month have a Poisson*$\left(\frac{1}{5}\right)$ *distribution. Assume also that when a claim occurs it has a Lognormal*$(100, 100)$ *distribution and that claims are independent. Simulate the total insurance claims over a 5 year period for 100 simulations.*

2.6 Financial Risk

Actuarial science is concerned with the application of probability to problems involving financial consequences. The risk of various outcomes has to be assessed in financial terms. The problem is usually to determine how much should be paid in order to be avoid or participate in a risk. In gambling the aim is to participate in risk in the hope of winning. In order to win it is necessary to put at risk some of your own money. In insurance a premium is paid in order to avoid the financial risk that will result from some contingent event such as death or a hailstorm. The question that needs to be answered is how much should be paid?

One approach is to determine the expected value of the payoff from a gamble or from a risk.

Example 2.15 *You are playing a game where a fair die is rolled only once. If a 1 appears face up then you lose \$100. If any other number appears you gain \$110. Calculate the expected pay-off from playing the game.*

Solution 2.15 *The expected pay-off is*

$$\frac{1}{6}(-100) + \frac{5}{6}(110) = 75$$

Now assume that in the example you are given the choice of paying \$75 in order to play the game. Would you pay this much to play? You would probably think twice about playing this game since you will effectively be \$175 worse off with a $\frac{1}{6}$ chance and \$35 better off with a $\frac{5}{6}$ chance.

Example 2.16 *Assume that you can buy a lottery ticket for \$1 with a 1 in 5,000,000 chance of winning \$1,000,000. What is the expected pay-off? Would you pay the \$1 to participate in the lottery?*

Solution 2.16 *The expected pay-off is \$0.20. For this you pay \$1. Most individuals would probably pay the \$1.*

It is clear from the above examples that using the expected value of a pay-off to place a value on risk is not necessarily the best way to make decisions about risk.

Example 2.17 *A dice is rolled for as many times as you wish to play the game. If the dice shows a 1,2, or 3 then you win twice the amount you have bet otherwise you win nothing. Consider the* doubling strategy *where you double your bet each time you lose and stop playing the game as soon as you win. You start by betting \$1. How much do you expect to win with the doubling strategy?*

Solution 2.17 *The winnings are* 2 *if you win on the first roll,* 2^2 *if you win on the second roll and* 2^n *if you win on the nth roll. The amount bet will be* 1 *for the first roll,* $1+2=3$ *for the second roll,* $1+2+4=7$ *for the third roll and* $1+2+2^2+\ldots+2^{n-1} = 2^n - 1$ *for the nth roll. Thus the expected winnings from the doubling strategy are* $\frac{1}{2}(2-1) + \left(\frac{1}{2}\right)^2(2^2-3) + \ldots + \left(\frac{1}{2}\right)^n(2^n - 2^n + 1) + \ldots = \sum_1^\infty \left(\frac{1}{2}\right)^i$. *This is equal to 1. In fact you are certain to double your money.*

In theory if we have a game with an even chance and we are able to find the money to ensure that we can always double our bet we can generate a sure profit from the doubling strategy. The closest game to this in a casino is the roulette wheel when we bet on either red or black. Unfortunately the casino has the benefit of the zero and so this strategy is not the sure win it might otherwise be!

Exercise 2.19 *Two fair dice are rolled separately. The first die is thrown as long as it shows a number less than or equal to 4. As soon as it shows a 5 or 6 it is no longer rolled. The second die is rolled as long as it shows a number greater than 1. As soon as a 1 is rolled it also is no longer rolled. You receive \$0.50 (50 cents) for each die rolled, thus if both dice are rolled you receive \$1, if only one die is rolled you receive \$0.50 and as soon as no dice are rolled you stop receiving anything. Calculate the expected payoff from playing this game?*

2.7 Conclusion

This chapter has covered the main concepts of probability used in actuarial science. The focus of actuarial science is on quantifying uncertainty and to do this it is necessary to use probability models. The main types of model required are those for survival probabilities (hazard rates) and those for payment amounts. The application of probability to actuarial problems will be covered in later chapters. A key concern in actuarial science is the probability that a financial security system will be able to meet its obligations. The probability that it will not be able to do this is referred to as the probability of ruin. The gamblers ruin problem was presented as an example of how these probabilities can be determined for games of chance.

2.8 Further Reading

There are many textbooks covering probability. A commonly used text is Rice [14]. A more advanced treatment is given in Ross [15].

2.9 Solutions to Exercises

Ex 2.1 *Number of ways is*

$$\frac{9!}{2} = 9.7.6.5.4.3 = 181,440$$

In Excel this can be evaluated using

$$\frac{\textit{FACT(9)}}{\textit{2}}$$

For a necklace both clockwise and anti-clockwise arrangements are the same (just turn the necklace over and you can not tell the difference). Thus we need to divide the number of ways of arranging 10 objects in a circle (9!) by 2.

Ex 2.2 *This is the same as selecting 3 subjects from 9 subjects. Answer*

$$\binom{9}{3} = \frac{9!}{6!3!} = \frac{9.8.7}{3.2} = 84$$

In Excel this can be evaluated using

$$\textit{COMBIN(9,3)}$$

Ex 2.3 *There are* $\binom{20}{10}$ *ways to select those seated at one of the tables. On each table* *the number of ways of arranging the seating is* 9!. *Answer is therefore*

$$\begin{aligned} & \binom{20}{10} 9!9! \\ = \; & \frac{20!}{10!10!} 9!9! \\ = \; & \frac{20!}{100} \\ = \; & 24,329,020,081,766,400 \end{aligned}$$

In Excel this can be evaluated as

FACT(20)/100

Ex 2.4

$$\begin{aligned} LHS &= \sum_{r=0}^{n} r \binom{n}{r} (1-p)^{n-r} (p)^{r} \\ &= \sum_{r=1}^{n} r \frac{n!}{r!\,(n-r)!} (1-p)^{n-r} (p)^{r} \\ &= \sum_{r=1}^{n} \frac{n!}{(r-1)!\,(n-r)!} (1-p)^{n-r} (p)^{r} \\ &= np \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!\,(n-r)!} (1-p)^{n-r} (p)^{r-1} \\ &= np \sum_{r=1}^{n} \binom{n-1}{r-1} (1-p)^{n-r} (p)^{r-1} \end{aligned}$$

Change variable to s *where*

$$r = s + 1$$

to get

$$\begin{aligned} &= np \sum_{s=0}^{n-1} \binom{n-1}{s} (1-p)^{n-(s+1)} (p)^{s} \\ &= np\,(1-p+p)^{n-1} \\ &= np \end{aligned}$$

Where we have used the Binomial Theorem.

Ex 2.5 *Just tedious algebra!*

Ex 2.6 *We need to evaluate all the possible cases and total those with sums between 7 and 13 i.e.* $8, 9, 10, 11,$ *and* 12. *These have been evaluated already in the text. Answer is*

$$\frac{21+25+27+27+25}{216} = \frac{125}{216}$$

Ex 2.7 *All can be heads with probability* $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ *and all can be tails with probability* $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$. *Required probability is* $\frac{1}{4}$. *This can be confirmed by evaluating all possible cases (there are 8 of them) and selecting the number of outcomes where all are heads or all are tails (there are two of them).*

Ex 2.8 *Probability of at least one ace in n draws = 1- probability of no aces in n draws. Probability of no aces in n draws =* $\left(\frac{12}{13}\right)^n$. *Required probability* $=1 - \left(\frac{12}{13}\right)^n$. *In order for this probability to exceed* $\frac{1}{2}$ *we require n such that*

$$1 - \left(\frac{12}{13}\right)^n > \frac{1}{2}$$

$$n > \frac{\log(\frac{1}{2})}{\log(\frac{12}{13})} = 8.66$$

Thus we need at least 9 draws.

Ex 2.9 *The expected value is*

$$\begin{aligned}
E[X] &= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} x \frac{\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=1}^{\infty} \lambda \frac{\lambda^{x-1}}{(x-1)!} \\
&= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}
\end{aligned}$$

Noting that

$$e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^x}{(x)!}$$

we obtain

$$\begin{aligned} E[X] &= \lambda e^{-\lambda} e^{\lambda} \\ &= \lambda \end{aligned}$$

The variance is

$$\begin{aligned} E[X-\mu]^2 &= E[X-\mu]^2 \\ &= \sum_{x=0}^{\infty} (x-\lambda)^2 \frac{e^{-\lambda}\lambda^x}{x!} \\ &= \sum_{x=0}^{\infty} \left(x^2 - 2x\lambda + \lambda^2\right) \frac{e^{-\lambda}\lambda^x}{x!} \\ &= \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!} - 2\lambda E[X] + \lambda^2 \\ &= \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!} - \lambda^2 \end{aligned}$$

Now

$$\begin{aligned} \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!} &= \sum_{x=1}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!} \\ &= \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{(x-1)!} \\ &= \lambda \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} \end{aligned}$$

Substituting $s = x - 1$ *we get*

$$\begin{aligned} &= \lambda \sum_{s=0}^{\infty} (s+1) \frac{e^{-\lambda}\lambda^s}{s!} \\ &= \lambda \left[\sum_{s=0}^{\infty} s \frac{e^{-\lambda}\lambda^s}{s!} + \sum_{s=0}^{\infty} \frac{e^{-\lambda}\lambda^s}{s!} \right] \\ &= \lambda [\lambda + 1] \\ &= \lambda^2 + \lambda \end{aligned}$$

Since $\sum_{s=0}^{\infty} s \frac{e^{-\lambda}\lambda^s}{s!} = E[S] = \lambda$ *and* $\sum_{s=0}^{\infty} \frac{e^{-\lambda}\lambda^s}{s!} = 1$ *by definition of the probability density (sum of all probabilities is one). Thus*

$$\begin{aligned} E[X-\mu]^2 &= \lambda^2 + \lambda - \lambda^2 \\ &= \lambda \end{aligned}$$

Ex 2.10 *Probability of 2 or more claims = 1 - Probability of zero or one claims. Required probability is*

$$\begin{aligned} & 1-e^{-\frac{1}{50}}-\frac{e^{-\frac{1}{50}}\frac{1}{50}}{1} \\ = & \ 1-0.980198673-0.019603973 \\ = & \ 0.000197353 \end{aligned}$$

Ex 2.11 *Variance of a Binomial random variable is*

$$\begin{aligned} E[X-\mu]^2 &= E\left[X^2-2X\mu+\mu^2\right] \\ &= E\left[X(X-1)+X-2X\mu+\mu^2\right] \\ &= E\left[X(X-1)+X(1-2\mu)+\mu^2\right] \\ &= \sum_{x=0}^{n}\left(x(x-1)+x(1-2np)+n^2p^2\right)\binom{n}{x}p^x(1-p)^{n-x} \\ &= \sum_{x=0}^{n}x(x-1)\binom{n}{x}p^x(1-p)^{n-x}+\sum_{x=0}^{n}x(1-2np)\binom{n}{x}p^x(1-p)^{n-x} \\ &\quad +\sum_{x=0}^{n}n^2p^2\binom{n}{x}p^x(1-p)^{n-x} \\ &= \sum_{x=0}^{n}x(x-1)\binom{n}{x}p^x(1-p)^{n-x}+(1-2np)np+n^2p^2 \\ &= \sum_{x=0}^{n}x(x-1)\binom{n}{x}p^x(1-p)^{n-x}+np-n^2p^2 \end{aligned}$$

Since $\sum_{x=0}^{n}x\binom{n}{x}p^x(1-p)^{n-x}=E[X]=np$ *and* $\sum_{x=0}^{n}\binom{n}{x}p^x(1-p)^{n-x}=1$. *Now*

$$\begin{aligned} \sum_{x=0}^{n}x(x-1)\binom{n}{x}p^x(1-p)^{n-x} &= \sum_{x=2}^{n}\frac{n!}{(x-2)!\,(n-x)!}p^x(1-p)^{n-x} \\ &= n(n-1)p^2\sum_{x=2}^{n}\frac{(n-2)!}{(x-2)!\,(n-x)!}p^{x-2}(1-p)^{n-x} \\ &= n(n-1)p^2\sum_{x=2}^{n}\binom{n-2}{x-2}p^{x-2}(1-p)^{n-x} \\ &= n(n-1)p^2 \end{aligned}$$

So that

$$\begin{aligned} E[X-\mu]^2 &= n(n-1)p^2+np-n^2p^2 \\ &= n^2p^2-np^2+np-n^2p^2 \\ &= -np^2+np \\ &= np(1-p) \end{aligned}$$

Ex 2.12 *This is a binomial probability. We want the probability of 3 successes from 6 trials where success is a 1 appearing on the dice. Required probability*

$$\begin{aligned}\binom{6}{3}\frac{1}{6}^3\frac{5}{6}^3 &= \frac{6.5.4}{3.2.1}\frac{1}{6}^3\frac{5}{6}^3 \\ &= 20\frac{1}{6}^3\frac{5}{6}^3 \\ &= 0.05358367\end{aligned}$$

In Excel this can be evaluated using

BINOMDIST(3,6,1/6,0)

Ex 2.13 *Probability of passing on the 4th attempt requires 3 failures then a pass. So probability is*

$$0.7^3 0.3 = 0.1029$$

Ex 2.14 *The probability that none of the 10 lightglobes will last 6 years is the probability that they all fail within 6 years. Probability that one globe fails in 6 years is*

$$F(6) = 1 - e^{-\frac{6}{3}} = 0.864665$$

The probability that all fail in ten years is

$$0.8646647^{10} = 0.233602$$

The probability that exactly 4 lightglobes will last 6 years is a binomial probability where a lightglobe lasting 6 years is a success (probability 1-0.8646647 = 0.135335) *and we require the probability of 4 successes from 10 trials. Required probability is*

$$\binom{10}{4}0.135335^4 0.8646647^6 = 0.0294406$$

In Excel this would be calculated using

BINOMDIST(4,10,0.135335,0)

Ex 2.15 *In Excel evaluate the probabilities at a range of discrete values for the binomial and the corresponding values for the normal. Then use Insert Chart to plot the cdfs.*

Ex 2.16 *Let* $X = \ln(1+R)$ *so that* $\ln(1+R)$ *has a normal*$\left(\ln 1.05, \sqrt{0.0004}\right)$ *distribution. Require*

$$\begin{aligned} \Pr[R < 0.04] &= \Pr[1 + R < 1.04] \\ &= \Pr[\ln(1+R) < \ln 1.04] \end{aligned}$$

We can express this in terms of a standard normal distribution by noting that $X = \mu + \sigma Z$ *where* Z *is standard normal. Thus* $\frac{X-\mu}{\sigma}$ *is standard normal. The probability can be written as*

$$\Pr\left[Z < \frac{\ln 1.04 - \ln 1.05}{\sqrt{0.0004}}\right] = \Pr[Z < -0.478473]$$

This can be evaluated in Excel using any of the following functions

$$NORMDIST(LN(1.04), LN(1.05), SQRT(0.0004), 1)$$

$$NORMSDIST(-0.478473)$$

or

$$LOGNORMDIST(1.04, LN(1.05), SQRT(0.0004))$$

to get an answer of 0.316157. This can also be evaluated using standard normal tables.

Ex 2.17 *Using the formula given in the text*

$$u_a = \left\{ \frac{1 - \left(\frac{q}{p}\right)^a}{1 - \left(\frac{q}{p}\right)^{a+b}} \right\}$$

we have in this case $a = 12$*,* $a + b = 24$*,* $p = \frac{1}{6}$*,* $q = \frac{5}{6}$*. The probability that* B *will end up with all the money is given by*

$$\begin{aligned} u_{12} &= \left\{ \frac{1 - (5)^{12}}{1 - (5)^{24}} \right\} \\ &= \left\{ 4.096 \times 10^{-9} \right\} \end{aligned}$$

i.e. 0.000000004 (very small).

Ex 2.18 *In Excel you need to use the Tools, Data Analysis, Random Number Generation tool. This allows you to generate Poisson and normal random numbers to be used in the simulation.*

Ex 2.19 *Note that the expected pay-off is simply the amount received per throw times the expected number of throws for each dice. The number of throws of each dice X has a geometric distribution. For the first dice we have $p = \frac{1}{3}$ and for the second dice we have $p = \frac{1}{6}$. For the geometric distribution we have $E[X] = \frac{1}{p}$. The expected pay-off is the sum of the expected pay-offs from each of the dice so that we get*

$$0.5(3 + 6) = \$4.50$$

Chapter 3
DEMOGRAPHY

3.1 Learning Objectives

The main objectives of this chapter are:

- to introduce the terminology of survival models and their use for modelling human survival,
- to introduce the terminology and concepts of the life table, and
- to introduce the mathematics of the life table for some simple mortality assumptions.

Demography is a separate discipline to actuarial science. We only cover some key concepts from demography that actuaries commonly use. More detailed coverage can be found in texts such as Brown (1993) [2].

3.2 Survival Models and Hazard Rates

Survival models are used in many areas including engineering, biostatistics, demography and actuarial science. The earliest development of a survival model used by actuaries was the life table based on population mortality, an example being that developed by John Graunt based on the Bills of Mortality of London. The early life insurance policies paid a sum insured on death of the insured life. These were initially short-term policies lasting for a year. They required information about the chance that a life of a given age would survive for a year. This was not readily available and usually the early insurance premium rates were too high.

Longer term insurance contracts were developed later. These were whole-of-life contracts which paid the sum insured at death in return for a level premium payable while the life was alive. We will consider the actuarial management of these insurance policies later. For now we will consider the models that can be used to determine the expected cash flows on these insurance policies.

It is difficult to predict the time of death of an individual life. In order to forecast the cash flows on an insurance policy we need to determine the expected number of deaths from a group of insured lives on future dates. In order to do this we need a model for the time-until-death random variable for the insured lives. We can develop a probability model that allows us to calculate the probability that the life will survive to specified ages. These models are called *survival models*. We then can use the models to determine expected numbers of deaths at future ages and hence the expected payments on insurance and annuity contracts.

3.2.1 Survival function

Consider the continuous random variable age-at-death X with distribution function $F_X(x)$. In insurance we are interested in the probability that a life survives to age x. Define the survival function for a new-born as

$$s(x) = \Pr(X > x) = 1 - F_X(x) \qquad x \geq 0$$

For a life aged 0 we have $F_X(0) = 0$ so that $s(0) = 1$. The probability density function is given by $F'_X(x) = f_X(x)$.

We are interested in the survival probability for a life that has survived to age x. This requires a *conditional probability*. A conditional probability is the probability that an event will happen given that another event has already happened. A conditional probability can be worked out as follows. Consider two events A and B. The probability that both events occur is equal to the probability that one occurs times the probability that the other occurs given that one has already occurred. Thus

$$\Pr(A \text{ and } B) = \Pr(A)\Pr(B|A) = \Pr(B)\Pr(A|B)$$

where $\Pr(A|B)$ is the probability that A occurs given that B has already occurred. We can write this as

$$\Pr(B|A) = \frac{\Pr(A \text{ and } B)}{\Pr(A)}$$

Thus for a life aged x the probability that the life will survive to age z $(z > x)$ will be given by the conditional probability

$$\Pr(X > z|X > x) = \frac{\Pr(X > z \text{ and } X > x)}{\Pr(X > x)} = \frac{s(z)}{s(x)}$$

Example 3.1 *If* $s(x) = 1 - \frac{x}{100}$ *for* $0 \leq x < 100$ *determine the probability that a life aged 20 will survive to age 65.*

Solution 3.1 *Required probability is* $\frac{s(65)}{s(20)} = \frac{1-\frac{65}{100}}{1-\frac{20}{100}} = : .437\,5$

Exercise 3.1 *If* $s(x) = \left(1 - \frac{x}{100}\right)^{\alpha}$ *for* $0 \leq x < 100 \quad \alpha > 0$ *calculate the probability that a life aged 30 will survive to age 75.*

The probability that a life aged x will die between the ages of y and z $(z > y > x)$ is

$$\begin{aligned}
\Pr\left(y < X \leq z | X > x\right) &= \frac{\Pr\left(X > y \text{ and } X \leq z\right)}{\Pr\left(X > x\right)} \\
&= \frac{\Pr\left(X > y\right) \text{-} \Pr\left(X > z\right)}{\Pr\left(X > x\right)} \\
&= \frac{s(y) - s(z)}{s(x)}
\end{aligned}$$

We define the *future lifetime* of a life aged x as $T\left(x\right) = X - x$. A life aged x is denoted by the symbol $\left(x\right)$.

Exercise 3.2 *If* $s\left(x\right) = \left(1 - \frac{x}{100}\right)^{\alpha}$ *for* $0 \leq x < 100 \quad \alpha > 0$ *calculate the probability that a life aged 30 will die prior to age 40.*

3.2.2 Actuarial notation

Actuaries use specialized notation for various survival and death probabilities. This notation is part of the International Actuarial Notation. The following notation is used

$$q_x = \Pr\left[\left(x\right) \text{ will die within a year}\right] = \Pr\left[T\left(x\right) \leq 1\right]$$

$$p_x = \Pr\left[\left(x\right) \text{ will survive at least a year}\right] = \Pr\left[T\left(x\right) > 1\right]$$

$${}_tq_x = \Pr\left[\left(x\right) \text{ will die within } t \text{ years}\right] = \Pr\left[T\left(x\right) \leq t\right] \qquad t \geq 0$$

$${}_tp_x = 1 -_t q_x = \Pr\left[\left(x\right) \text{ will survive at least } t \text{ years}\right] = \Pr\left[T\left(x\right) > t\right]$$

Note that $q_x + p_x = 1$.

We can write these probabilities in terms of the survival function as follows:

$${}_xp_0 = s(x)$$

$${}_tp_x = \frac{{}_{x+t}p_0}{{}_xp_0} = \frac{s(x+t)}{s(x)}$$

$$ {}_tq_x = 1 - \frac{s(x+t)}{s(x)} $$

In insurance we are also interested in events such as a life aged x surviving t years and dying in the next u years. We need these probabilities for deferred insurances and annuities which are contracts that pay an amount conditional on survival for a specified number of years. The actuarial notation for the probability that a life (x) will survive t years and die in the next u years is ${}_{t|u}q_x$. We have

$$
\begin{aligned}
{}_{t|u}q_x &= \Pr\left[t < T(x) \le t+u\right] \\
&= \frac{s(x+t) - s(x+t+u)}{s(x)} \\
&= \left[1 - \frac{s(x+t+u)}{s(x)}\right] - \left[1 - \frac{s(x+t)}{s(x)}\right] \\
&= {}_{t+u}q_x - {}_t\, q_x
\end{aligned}
$$

Example 3.2 *If* $s(x) = \left(1 - \frac{x}{100}\right)^2$ *for* $0 \le x \le 100$ *calculate* ${}_5p_{35}$.

Solution 3.2 ${}_5p_{35} = \frac{s(40)}{s(35)} = \frac{\left(1-\frac{40}{100}\right)^2}{\left(1-\frac{35}{100}\right)^2} = 0.85207.$

Exercise 3.3 *If* $s(x) = \exp\left[-\frac{B}{\ln c}\left(c^x - 1\right)\right]$ *derive an expression for* ${}_{t|u}q_x$.

Exercise 3.4 *Show that* ${}_{t|u}q_x = {}_t\, p_x \;{}_uq_{x+t}$

3.2.3 Hazard rates

Survival models can be characterised by their *hazard rate or failure rate function.* It is interesting to note that the word *hazard* may have originated from *al-zhar* the Arabic name for die ([4], p34). In actuarial science and demography the *hazard or failure rate* is referred to as the *force of mortality.* We define the hazard rate as

$$
\begin{aligned}
\mu(x) &= \frac{-\frac{d}{dx}s(x)}{s(x)} \\
&= \frac{-s'(x)}{s(x)} \\
&= \frac{f_X(x)}{1 - F_X(x)}
\end{aligned}
$$

In actuarial notation the force of mortality is denoted by μ_x. This is actually a conditional probability density function. It gives the probability density of the age-at-death (X) given that the life has survived to age x.

Note that

$$\frac{-\frac{d}{dy}s\left(y\right)}{s\left(y\right)}=-\frac{d}{dy}\ln s\left(y\right)$$

so that

$$-\mu\left(y\right)dy=d\ln s\left(y\right)$$

Integrating from x to $x+t$ we have

$$\int_{x}^{x+t}-\mu\left(y\right)dy=\int_{x}^{x+t}d\ln s\left(y\right)=\ln\left[\frac{s\left(x+t\right)}{s\left(x\right)}\right]$$

Thus

$$\frac{s\left(x+t\right)}{s\left(x\right)}={}_t\,p_x=\exp\left[\int_{x}^{x+t}-\mu\left(y\right)dy\right]$$

We also have

$$s(x)={}_x\,p_0=\exp\left[\int_{0}^{x}-\mu\left(y\right)dy\right]$$

We can write the distribution and density function of the age-at-death random variable X in terms of the hazard rate as

$$F_X(x)=1-s(x)=1-\exp\left[\int_{0}^{x}-\mu\left(y\right)dy\right]$$

$$f_X\left(x\right)=\frac{d}{dx}F_X(x)=\mu\left(x\right)\exp\left[\int_{0}^{x}-\mu\left(y\right)dy\right]=\mu\left(x\right)s(x)$$

We can also write the distribution and density function of the future lifetime random variable $T\left(x\right)$ as

$$F_{T(x)}(t)=1-\frac{s(x+t)}{s(x)}=1-\exp\left[\int_{x}^{x+t}-\mu\left(y\right)dy\right]$$

$$f_{T(x)}(t)=\frac{d}{dt}F_{T(x)}(t)=\mu\left(x+t\right)\exp\left[\int_{x}^{x+t}-\mu\left(y\right)dy\right]=\mu\left(x+t\right){}_t\,p_x$$

Example 3.3 *Evaluate* $\int_0^\infty \mu\left(x\right)s(x)dx$.

Solution 3.3 *Since $\mu(x)\, s(x)$ is a probability density function we know that the integral is equal to 1. This is also the probability that a new born will die between the ages of 0 and ∞, which is of course 1.*

Exercise 3.5 *Derive an expression for $f_X(x)$ for $s(x) = e^{-\frac{x^2}{10}} \quad x \geq 0$.*

Exercise 3.6 *If $\mu(x+t) = t \quad t \geq 0$ determine $F_X(x)$.*

Exercise 3.7 *Say the force of mortality for a group of lives is multiplied by a constant k. Show that the new survival function for this group of lives will be $({}_xp_0)^k$. This is called a proportional hazard.*

3.3 The Life Table

Human survivorship depends on many factors. Empirically hazard rates depend on age and sex for human lives. Other factors are also important such as whether the person is a smoker or a non-smoker, or engages in hazardous sports such as parachute jumping.

Data has been collected on populations through the census and for deaths for many years in most countries. This data is used to determine a *life table* for population mortality rates by age. To calculate mortality rates the census is used to determine the number of people alive in the population for different ages. Records of deaths are used to calculate the number of deaths at the different ages. Death rates at different ages and for different sexes can then be calculated.

Insurance companies collect data on claims for their policies and the number of policies and sum insured for different ages. This data is collected by The Institute of Actuaries of Australia from many of the Australian companies. The data is collected for different types of insurance policies such as life annuities, term insurance and whole-of-life insurance. The data from these insurance companies is then pooled and used to construct *life tables* for insured lives and annuitants.

The *life table* is used in actuarial practice for insurance and annuity calculations. Insurance companies require medical statements and other evidence of health before issuing a life insurance policy. This results in the healthier lives taking out insurance. As a result insured lives mortality is lower than for the population.

The detail as to how the life table is constructed is the subject of *actuarial statistics.*

The life table is a table showing for each age x:

- q_x the probability of death between age x and $x+1$,
- d_x the (expected) number of deaths aged x last birthday,
- l_x the (expected) number alive at exact age x,

- L_x the (expected) number living aged between x and $x+1$ (x last birthday), also the (expected) number of years lived by the l_x lives aged x over the year from age x to $x+1$,

- T_x the number of lives aged x or greater, also the (expected) total future lifetime of the l_x lives aged x, and

- $\mathring{e}_x$ the average future lifetime of a life aged x, also referred to as the *complete expectation of life.*

Note that L_x is a measure of the *exposed-to-risk* of dying aged x last birthday. Any life in this group who dies will be classified as a death at age x. We can see that

$$L_x = \int_0^1 l_{x+t} dt \approx \frac{l_x + l_{x+1}}{2}$$

Note also that by definition

$$T_x = \sum_{t=0}^{\infty} L_{x+t}$$

Now T_x is the total number of future years lived by the l_x lives aged x so that the average number of years lived is $\frac{T_x}{l_x}$.

The figures provide a sample of some Australian life tables.

The first table is the IA64-70 Table for Australian Insured Lives based on the mortality of whole-of-life (without term riders) policies for mainly male lives from data contributed by 14 Life Offices. In case you are wondering, a term rider is an addition to the basic insurance policy in the form of a term insurance. The data used was for the period January 1964 to December 1970. The table was published by The Institute of Actuaries of Australia. Since insured lives are initially selected by the insurance companies the mortality of a life aged x who has just purchased an insurance policy is lighter than the mortality of a life now aged x who purchased a life insurance policy in the past. For this reason life insurance mortality tables usually determine the mortality rate depending on the time since the life took out the policy as well as age. Tables with mortality rates by both age and duration are referred to as *select tables.* The tables usually assume that after a certain time period the effect of any selection will have diminished so that it can be ignored and mortality rates need only differentiate by age after this period. This period is called the *select period.*

The mortality rates for lives who have owned a policy for longer than the select period is referred to as *ultimate mortality.* For the IA64-70 Table the select period was 2 years. The table shows only the ultimate mortality rates.

The other table shown is for Australian males. It is a population mortality table based on all male lives in Australia in the 1991 census and male deaths over the period 1990 to 1992. The death rates are determined as the average deaths of

IA64-70 Ultimate (mainly male insured lives)

Age	qx	dx	lx	Lx	Tx	e0x
10	0.00034	340.0	999999	999829	63241421	63.24
11	0.00035	349.9	999659	999484	62241592	62.26
12	0.00038	379.7	999309	999119	61242108	61.28
13	0.00043	429.5	998929	998715	60242989	60.31
14	0.00053	529.2	998500	998235	59244274	59.33
15	0.00070	698.6	997971	997621	58246039	58.36
16	0.00094	937.4	997272	996803	57248418	57.41
17	0.00123	1225.5	996335	995722	56251615	56.46
18	0.00156	1552.4	995109	994333	55255893	55.53
19	0.00184	1828.1	993557	992643	54261560	54.61
20	0.00192	1904.1	991729	990777	53268917	53.71
21	0.00181	1791.6	989824	988929	52278140	52.82
22	0.00160	1580.9	988033	987242	51289212	51.91
23	0.00138	1361.3	986452	985771	50301969	50.99
24	0.00118	1162.4	985091	984510	49316198	50.06
25	0.00107	1052.8	983928	983402	48331688	49.12
26	0.00103	1012.4	982876	982369	47348286	48.17
27	0.00101	991.7	981863	981367	46365917	47.22
28	0.00100	980.9	980872	980381	45384550	46.27
29	0.00101	989.7	979891	979396	44404169	45.32
30	0.00103	1008.3	978901	978397	43424773	44.36
31	0.00105	1026.8	977893	977379	42446376	43.41
32	0.00109	1064.8	976866	976333	41468997	42.45
33	0.00113	1102.7	975801	975250	40492663	41.50
34	0.00117	1140.4	974698	974128	39517413	40.54
35	0.00123	1197.5	973558	972959	38543285	39.59
36	0.00130	1264.1	972361	971729	37570326	38.64
37	0.00139	1349.8	971097	970422	36598597	37.69
38	0.00149	1444.9	969747	969024	35628176	36.74
39	0.00160	1549.3	968302	967527	34659152	35.79
40	0.00174	1682.1	966752	965911	33691624	34.85
41	0.00190	1833.6	965070	964154	32725713	33.91
42	0.00209	2013.2	963237	962230	31761559	32.97
43	0.00230	2210.8	961224	960118	30799329	32.04
44	0.00254	2435.9	959013	957795	29839211	31.11
45	0.00282	2697.5	956577	955228	28881416	30.19
46	0.00314	2995.2	953879	952382	27926188	29.28
47	0.00350	3328.1	950884	949220	26973807	28.37
48	0.00391	3704.9	947556	945704	26024587	27.46
49	0.00437	4124.6	943851	941789	25078883	26.57
50	0.00489	4595.3	939726	937429	24137094	25.69

Figure 1

A90-92 Males - Australian Population

Age	qx	dx	lx	Lx	Tx	e0x
0	0.00814	8140.0	999999	995929	74319391	74.32
1	0.00064	634.8	991859	991542	73323462	73.93
2	0.00047	465.9	991224	990991	72331920	72.97
3	0.00034	336.9	990758	990590	71340929	72.01
4	0.00025	247.6	990421	990298	70350339	71.03
5	0.00021	207.9	990174	990070	69360042	70.05
6	0.00020	198.0	989966	989867	68369972	69.06
7	0.00019	188.1	989768	989674	67380105	68.08
8	0.00018	178.1	989580	989491	66390431	67.09
9	0.00018	178.1	989402	989313	65400940	66.10
10	0.00018	178.1	989224	989135	64411627	65.11
11	0.00017	168.1	989046	988962	63422493	64.12
12	0.00018	178.0	988877	988788	62433531	63.14
13	0.00022	217.5	988699	988591	61444743	62.15
14	0.00030	296.5	988482	988334	60456152	61.16
15	0.00044	434.8	988185	987968	59467818	60.18
16	0.00063	622.3	987751	987439	58479850	59.21
17	0.00084	829.2	987128	986714	57492411	58.24
18	0.00103	1015.9	986299	985791	56505697	57.29
19	0.00117	1152.8	985283	984707	55519906	56.35
20	0.00126	1240.0	984130	983510	54535199	55.41
21	0.00130	1277.8	982890	982252	53551688	54.48
22	0.00131	1285.9	981613	980970	52569437	53.55
23	0.00130	1274.4	980327	979690	51588467	52.62
24	0.00130	1272.8	979052	978416	50608777	51.69
25	0.00130	1271.1	977780	977144	49630361	50.76
26	0.00130	1269.5	976509	975874	48653217	49.82
27	0.00130	1267.8	975239	974605	47677344	48.89
28	0.00130	1266.2	973971	973338	46702738	47.95
29	0.00130	1264.5	972705	972073	45729400	47.01
30	0.00131	1272.6	971441	970804	44757327	46.07
31	0.00132	1280.6	970168	969528	43786523	45.13
32	0.00133	1288.6	968887	968243	42816996	44.19
33	0.00135	1306.3	967599	966946	41848753	43.25
34	0.00138	1333.5	966292	965626	40881807	42.31
35	0.00142	1370.2	964959	964274	39916181	41.37
36	0.00147	1416.5	963589	962880	38951907	40.42
37	0.00152	1462.5	962172	961441	37989027	39.48
38	0.00159	1527.5	960710	959946	37027586	38.54
39	0.00167	1601.8	959182	958381	36067640	37.60
40	0.00177	1694.9	957580	956733	35109259	36.66
41	0.00188	1797.1	955885	954987	34152526	35.73
42	0.00201	1917.7	954088	953130	33197539	34.80
43	0.00217	2066.2	952171	951138	32244409	33.86
44	0.00235	2232.7	950104	948988	31293272	32.94
45	0.00256	2426.6	947872	946658	30344283	32.01
46	0.00281	2656.7	945445	944117	29397625	31.09
47	0.00309	2913.2	942788	941332	28453508	30.18
48	0.00340	3195.6	939875	938277	27512176	29.27
49	0.00377	3531.3	936680	934914	26573899	28.37
50	0.00418	3900.6	933148	931198	25638985	27.48
51	0.00464	4311.7	929248	927092	24707787	26.59
52	0.00517	4781.9	924936	922545	23780695	25.71
53	0.00577	5309.3	920154	917500	22858149	24.84
54	0.00644	5891.6	914845	911899	21940650	23.98
55	0.00720	6544.5	908953	905681	21028751	23.14
56	0.00804	7255.4	902409	898781	20123070	22.30
57	0.00897	8029.5	895154	891139	19224288	21.48
58	0.01001	8880.1	887124	882684	18333150	20.67
59	0.01115	9792.4	878244	873348	17450466	19.87
60	0.01241	10777.5	868451	863063	16577118	19.09

Figure 2

males between 1990-92 divided by the 1991 census figures for each age. These rates are smoothed using an actuarial technique called *graduation*.

The figures in these mortality tables were constructed as follows. The insurance company and population census and deaths data provides the information to calculate the q_x rates. Once these are calculated the table starts with a *radix* which is the base for the table. It is simply the number of individuals at the earliest age of the table which is usually age 0. In the two Australian life tables the radix is $1,000,000$. For the population mortality table (A90-92 males) the radix is $l_0 = 1,000,000$. In the case of the IA64-70 table the radix chosen was $l_{10} = 1,000,000$. The rest of the table is constructed as follows:

$$d_x = q_x l_x$$

$$l_{x+1} = l_x - d_x$$

$$L_x = \frac{l_x + l_{x+1}}{2}$$

$$T_x = T_{x+1} + L_x$$

$$\mathring{e}_x = \frac{T_x}{l_x}$$

We should note that the life table is normally interpreted as a *deterministic model* of survival. This means that the l_x values are interpreted as the number of lives exact age x surviving from the initial l_0 lives. For any group of lives we know that we cannot predict exactly their time of death and that this should be treated as a random variable. If we consider a group of l_0 new borns then the probability that any one of these will survive to age x is $s(x)$. Assuming the lives are independent then the number of survivors to age x will have a binomial distribution with $n = l_0$ and $p = s(x)$ since each life can either survive to age x with probability p or die before age x with probability $(1 - p)$. The expected number of lives out of the initial l_0 lives who survive to exact age x will be given by the expected value of the binomial distribution, therefore

$$l_x = l_0 s(x)$$

The variance of the number of lives surviving to age x will be

$$l_0 s(x) \left[1 - s\left(x\right)\right]$$

We note also that

$${}_t p_x = \frac{s\left(x+t\right)}{s\left(x\right)} = \frac{l_0 s\left(x+t\right)}{l_0 s\left(x\right)} = \frac{l_{x+t}}{l_x}$$

Example 3.4 *Use the IA64-70 Life Table to calculate*

- *the expected age at death of a life aged 20*
- *the probability that a life age 20 will survive to age 40*
- *the probability that a life aged 20 will die within the next 10 years.*

Solution 3.4 *The answers are*

- $20 + \mathring{e}_{20} = 20 + 53.71 = 73.\,71.$
- ${}_{20}p_{20} = \frac{l_{40}}{l_{20}} = \frac{966752}{991729} = .\,974\,81$
- ${}_{10}q_{20} = \frac{l_{20}-l_{30}}{l_{20}} = \frac{991729-978901}{991729} = :\ 1.\,293\,5 \times 10^{-2}$

Exercise 3.8 *Use the A90-92 Life Table to calculate*

- *the expected age at death of a life aged 20*
- *the probability that a life age 20 will survive to age 40*
- *the probability that a life aged 20 will die within the next 10 years.*

3.4 Laws of Mortality

A number of different models have been proposed for the hazard rate for human lives. These are sometimes referred to as *laws of mortality* since they are based on an assumption as to how mortality is affected by age. These allow simple analytical calculations to be performed in place of the numerical calculations required using a life table. Over a range of ages some of these models can work reasonably well as a fit to human life tables. We cover a few simple cases.

3.4.1 DeMoivre's Law

In 1725 Abraham de Moivre performed some early actuarial calculations based on the assumption that the number alive according to the life table decreased in arithmetical progression. In this case l_x is a linear function of x. We can write this assumption in terms of the survival function as

$$s\,(x) = 1 - \frac{x}{\omega} \quad 0 \leq x < \omega$$

We have a hazard rate (force of mortality) given by

$$\mu(x) = \frac{-s'\,(x)}{s\,(x)} = \frac{1}{\omega - x} \quad 0 \leq x < \omega$$

3.4.2 Gompertz Law

Benjamin Gompertz in 1825 hypothesized that the force of mortality increases with age in geometrical progression. The hazard rate would then be given by

$$\mu(x) = Bc^x \quad B > 0,\ c > 1,\ x \geq 0$$

Thus as you get older your chance of death increases more and more rapidly. We have

$$\begin{aligned} s(x) &= \exp\left[\int_0^x -\mu(y)\,dy\right] \\ &= \exp\left[\int_0^x -Bc^y dy\right] \\ &= \exp\left[-\frac{B}{\ln c}c^y\right]_0^x \\ &= \exp\left[-\frac{B}{\ln c}(c^x - 1)\right] \end{aligned}$$

There are various ways of determining the values to use for the *parameters* of the survival function. The parameters of the Gompertz curve are B and c. A commonly used technique in statistics is to use *least squares*. Least squares selects the numerical values of the parameters by minimizing the sum of the squared differences between the fitted values and the actual values over the range of ages to be used for fitting the function. Thus if the actual values for $s(x)$ derived from the life table are denoted by $s^a(x)$, then least squares will fit the Gompertz curve over the age range x_l to x_u by selecting values of B and c that minimize the function

$$\sum_{x_l}^{x_u}\left[s^a(x) - \exp\left[-\frac{B}{\ln c}(c^x - 1)\right]\right]^2$$

This is readily done in a spreadsheet such as Excel using the Solver.

When fitted to the Australian IA64-70 Life Table over the ages 10 to 110 the best fit Gompertz curve using least squares was given by

$$\mu(x) = 0.000046\,(1.100837)^x$$

However the fit of this curve to the higher and younger ages is not so good. It is possible to limit the range of ages used so that the Gompertz curve provides a better fit. For instance, from ages of 30 up to around age 80 the Gompertz curve provides a reasonable fit to the IA64-70 data.

3.4.3 Makeham's Law

In 1867 Makeham suggested an addition of a constant to account for accidents and infections as well as an increase in hazard geometrically with age. The form of this hazard function is

$$\mu(x) = A + Bc^x \quad A > -B,\, B > 0,\; c > 1,\, x \geq 0$$

When fitted to the IA64-70 Life Table over the ages 10 to 110 the best fit Makeham curve is

$$\mu(x) = 0.0003225 + 0.000031\,(1.1066413)^x$$

This curve does not fit much better than the Gompertz curve but still provides a reasonable fit for a limited range of ages up to around age 80.

Many other laws of mortality have been proposed which provide better fits to the life table data actually observed.

Exercise 3.9 *Use the data from a Population Life Table and fit the parameters of the Gompertz and Makeham hazard function using least squares. You may wish to use the Solver in Excel or a least squares fitting procedure in a package such as Matlab, Minitab or Maple. Plot the fitted and actual values for q_x on the same graph and comment on your results.*

3.5 Conclusion

In this chapter we have introduced the concept of a survival model and related it to one of the earliest tool used in actuarial science, the life table. We introduced the notation used by actuaries and demographers and showed how the life table can be used to calculate probabilities of survival and death.

3.6 Solutions to Exercises

Ex 3.1 *Required probability*

$$\begin{aligned}
\frac{s\,(75)}{s\,(30)} &= \frac{\left(1-\frac{75}{100}\right)^{\alpha}}{\left(1-\frac{30}{100}\right)^{\alpha}} \\
&= \left(\frac{0.25}{0.7}\right)^{\alpha} \\
&= (0.35714)^{\alpha}
\end{aligned}$$

Ex 3.2 *Required Probability*

$$\begin{aligned}
\frac{s\,(30)-s\,(40)}{s\,(30)} &= \frac{\left(1-\frac{30}{100}\right)^{\alpha}-\left(1-\frac{40}{100}\right)^{\alpha}}{\left(1-\frac{30}{100}\right)^{\alpha}} \\
&= 1-\left(\frac{0.6}{0.7}\right)^{\alpha}
\end{aligned}$$

Ex 3.3

$$\begin{aligned} {}_{t|u}q_x &= \frac{s(x+t) - s(x+t+u)}{s(x)} \\ &= \frac{\exp\left[-\frac{B}{\ln c}\left(c^{x+t}-1\right)\right] - \exp\left[-\frac{B}{\ln c}\left(c^{x+t+u}-1\right)\right]}{\exp\left[-\frac{B}{\ln c}\left(c^{x}-1\right)\right]} \\ &= \frac{\exp\left[-\frac{B}{\ln c}\left(c^{x+t}\right)\right] - \exp\left[-\frac{B}{\ln c}\left(c^{x+t+u}\right)\right]}{\exp\left[-\frac{B}{\ln c}\left(c^{x}\right)\right]} \\ &= \exp\left[-\frac{B}{\ln c}\left(c^{t}\right)\right] - \exp\left[-\frac{B}{\ln c}\left(c^{t+u}\right)\right] \end{aligned}$$

Ex 3.4

$$\begin{aligned} RHS &= {}_{t|u}q_x \\ &= \frac{s(x+t) - s(x+t+u)}{s(x)} \\ &= \frac{s(x+t)}{s(x)}\left[1 - \frac{s(x+t+u)}{s(x+t)}\right] \\ &= {}_tp_x\left[1 -_u p_{x+t}\right] \\ &= {}_tp_x\left[{}_uq_{x+t}\right] \end{aligned}$$

Ex 3.5 *By definition we have*

$$F_X(x) = 1 - s(x)$$

so that

$$\begin{aligned} f_X(x) &= \frac{d}{dx}F_X(x) \\ &= -\frac{d}{dx}s(x) \end{aligned}$$

If $s(x) = e^{-\frac{x^2}{10}}$ *then*

$$f_X(x) = \frac{x}{5}e^{-\frac{x^2}{10}}$$

Ex 3.6 *We have*

$$\begin{aligned} F_X(x) &= 1 - s(x) \\ &= 1 - \exp\left[\int_0^x -\mu(y)\,dy\right] \\ &= 1 - \exp\left[\int_{-x}^0 -\mu(x+t)\,dt\right] \end{aligned}$$

Thus for $\mu(x+t) = t$ we have

$$\begin{aligned} F_X(x) &= 1 - \exp\left[\int_{-x}^{0} -tdt\right] \\ &= 1 - \exp\left[\frac{-t^2}{2}|_{-x}^{0}\right] \\ &= 1 - \exp\left[\frac{x^2}{2}\right] \end{aligned}$$

Ex 3.7 *We have*

$$s(x) =_x p_0 = \exp\left[\int_0^x -\mu(y)\,dy\right]$$

Now if the force of mortality is multiplied by k we get

$$\begin{aligned} s(x) &= \exp\left[\int_0^x -k\mu(y)\,dy\right] \\ &= \left[\exp\left[\int_0^x -\mu(y)\,dy\right]\right]^k \\ &= \left({}_xp_0\right)^k \end{aligned}$$

Ex 3.8 *Using A90-92 we have the expected age at death of a life aged 20*

$$\begin{aligned} & 20 + \mathring{e}_{20} \\ =& \ 20 + 55.45 \\ =& \ 75.45 \end{aligned}$$

The probability that a life age 20 will survive to age 40

$$\begin{aligned} {}_{20}p_{20} &= \frac{l_{40}}{l_{20}} \\ &= \frac{957580}{984130} \\ &= 0.97302 \end{aligned}$$

The probability that a life aged 20 will die within the next 10 years

$$\begin{aligned} {}_{10}q_{20} &= 1 -_{10} p_{20} \\ &= 1 - \frac{l_{30}}{l_{20}} \\ &= 1 - \frac{971441}{984130} \\ &= 0.012894 \end{aligned}$$

Ex 3.9 *You will need to numerically determine a minimum of the sum of the squared differences between the actual* q_x *rates and those determined using the Gompertz and Makeham hazard function formula. Take care if using the Solver in Excel. Make sure that you have found a minimum.*

Chapter 4
HIGH FINANCE

4.1 Learning Objectives

The main objectives of this chapter are:

- to introduce the mathematics of finance required to value known future cash-flows,
- to introduce the concept of an annuity, a life annuity and the insurance mathematics required to value these annuities, and
- to introduce the basic types of securities and their characteristics and the investment management issues of concern to actuaries.

4.2 Compound Interest

4.2.1 Introduction

One of the most fundamental tools of actuarial science is the mathematics of compound interest. The early actuarial journals of the 1800's contain many scientific articles developing the mathematics of finance and the application of the mathematics of finance to insurance problems.

The valuation of *contingent cash flows* is a major application in actuarial science of financial and insurance mathematics. These contingent cash flows depend on the survival or death of a life and they will also usually depend on financial market returns. There are many insurance products where the payment of benefits on death or survival depend on the future value of assets held by the insurance fund. Many insurance products have guaranteed minimum death benefits and also guaranteed maturity values. These guarantees are contingent on the value of the assets of the insurance fund. Superannuation annuities and other benefits are also often contingent on the level of inflation or returns in financial markets.

Before considering the basic ideas underlying valuation of contingent cash flows we need to understand how to value fixed and know cash flows that are not contingent on any future uncertainty. We will start by considering a borrowing of a fixed amount of money, usually referred to as the *principal, capital or loan amount*, repayable at a future specified date, the *maturity date* of the loan. Some loans include payment of principal in the cash flow paid during the life of the loan. For an interest only loan, no repayment of principal is assumed to occur before the maturity date. If an amount is borrowed then an amount of interest is payable by the borrower to the lender to compensate the lender for the use of the funds. Interest has been charged on loans from at least the Middle Ages. In some countries the charging of interest is considered to be a crime. The charging of excessive interest is called *usury.*

4.2.2 Definitions

The interest rate charged reflects the *market rate* of interest for the maturity of the loan. These interest rates vary with the *term or tenor* of the loan. The *term or tenor* of the loan is the time to maturity of the loan. Interest is paid at different *frequencies* during the year for different loans. Interest periods are often monthly, quarterly, semi-annual or annual. Interest paid at the end of the period is payable *in arrears* and interest payable at the start of the period is payable *in advance.* Interest rates can be *fixed* throughout the term of the loan, or *variable. A variable rate* loan is also referred to as a *floating rate* loan.

Assume that the initial amount of the loan is L and that the loan is repaid in T years time with interest paid m times per year. Thus the number of periods of the loan is mT. Let $V(j)$ be the amount of the loan at the end of time period j $(j = 1, 2, \ldots mT)$, immediately **after** any cash flow due at time j has occurred. Note that time period j covers the period from time $j-1$ to j. In general we assume a floating interest rate. The interest rate that applies for time period j is set at the start of the time period, at time $j-1$, and is denoted by $r(j-1)$ for $j = 1, 2 \ldots mT$. A fixed rate loan is a special case with $r(j-1) = r$, a constant for all time periods, which is set at the start of the loan.

4.2.3 Effective interest rates

Note that this interest rate is **per dollar** per time period not per cent per annum. This rate is referred to as an *effective per period* interest rate. In financial markets, interest rates are normally quoted as *per annum percentage nominal rates* assuming a specified number of periods per year. The number of periods, denoted by m, is also called the *compounding frequency* of the interest rate. A nominal rate is an annual rate for which the corresponding per period rate is derived by dividing the rate by the number of periods m in a year.

Example 4.1 *A 10-year loan is payable with quarterly cash flows and with a constant nominal interest rate of 12% p.a payable quarterly. Calculate* $r(j)$ *for this loan.*

Solution 4.1 *We have* $T = 10$, $m = 4$, *and* $r(j) = r$ *for all* j *with* $r = \frac{12}{400} = 0.03$.

In what follows we will use an *effective interest rate per period.* This period could be monthly, quarterly, semi-annual or annual. In general we will have m periods per annum with $m = 12$ corresponding to monthly, $m = 4$ corresponding to quarterly and so on. If we have a nominal annual rate then we can convert this to an effective per period rate by dividing by the number of periods. Thus if $j^{(m)}$ is the per annum nominal interest rate with $m$thly periods, assumed to be per dollar so that it is the percentage rate divided by 100, then r the *per period effective rate* is

$$r = \frac{j^{(m)}}{m}$$

Although it is normal practice in financial markets to quote interest rates as nominal rates we often wish to convert nominal rates into effective rates. For example, say we have an annual nominal rate assuming quarterly compounding (time periods) and we wish to value annual cash flows using an equivalent effective rate. In this case we need to convert the nominal quarterly rate to an annual effective rate. We do this by compounding up the quarterly nominal rate for a full year. Thus for an annual effective rate we have

$$1 + j = \left(1 + \frac{j^{(m)}}{m}\right)^m$$

where j is the annual effective interest rate and $j^{(m)}$ is the annual nominal interest rate assuming $m$thly periods.

Example 4.2 *Calculate the annual effective rate corresponding to 6% p.a. nominal assuming a compounding frequency of monthly and semi-annual.*

Solution 4.2 *For the monthly case the annual effective interest rate is*

$$\begin{aligned} j &= \left(1 + \frac{0.06}{12}\right)^{12} - 1 \\ &= 0.061678 \end{aligned}$$

or 6.1678% p.a. effective. For the semi-annual case we have

$$\begin{aligned} j &= \left(1 + \frac{0.06}{2}\right)^{2} - 1 \\ &= 0.0609 \end{aligned}$$

or 6.09% p.a. effective. Note that for a fixed nominal rate, the more frequently that the interest rate compounds, the higher the effective p.a. interest rate.

Exercise 4.1 *Calculate the nominal annual rate of interest assuming quarterly compounding equivalent to an annual effective interest rate of 7.5% p.a.*

4.2.4 Recurrence relations

Denote the cash flows on the loan at time j $(j = 0, 1, 2 \ldots mT)$ by $C(j)$. Note that we will treat payments of principal and interest on a loan as positive amounts so that usually $C(j) \geq 0$. Note that $V(0)$ is **after** any cash flow at time 0 so that if there is a cash flow at time 0 then $V(0)$ will be net of this payment.

Consider the loan at time t and what happens to the loan over a single time period. The amount of the loan will increase by the interest rate for that period and will decrease by the cash flow paid at the end of the time period. Thus we can write

$$V(j+1) = V(j)\left[1 + r(j)\right] - C(j+1) \quad \text{for } j = 0 \text{ to } mT - 1$$

Note that this is a difference equation and we can rewrite it as

$$\begin{aligned} V(j+1) - V(j) &= V(j)\left[r(j)\right] - C(j+1) \quad \text{for } j = 0 \text{ to } mT - 1 \\ \Delta V(j) &= V(j)\left[r(j)\right] - C(j+1) \quad \text{for } j = 0 \text{ to } mT - 1 \end{aligned}$$

where $\Delta V(j) = V(j+1) - V(j)$. This is also called a *forward recurrence relation.*

Remark 4.1 *If we assume that payments are made in continuous time at an annual rate of $C(t)$ at time t and that interest accrues continuously at an annual rate $\delta(t)$ at time t then this difference equation can be written as a differential equation as follows*

$$dV(t) = V(t)\left[\delta(t)\right]dt - C(t)dt$$

which becomes

$$\frac{dV(t)}{dt} = V(t)\left[\delta(t)\right] - C(t)$$

At the end of the loan the amount of the loan outstanding should be zero by definition. Therefore

$$V(mT) = 0$$

We also have that

$$V(0) = L - C(0)$$

We can now solve for $V(t)$ either by starting at the terminal condition with $V(mT) = 0$ and working backwards or starting at the initial condition $V(0) = L - C(0)$ and working forwards.

If we consider the initial condition and work forwards then over the first period of the loan we have

$$V(1) = V(0)\left[1 + r(0)\right] - C(1)$$

Since $V(0) = L - C(0)$ we have

$$V(1) = \{L - C(0)\}[1 + r(0)] - C(1)$$

For the next time period we have

$$\begin{aligned} V(2) &= V(1)[1 + r(1)] - C(2) \\ &= [\{L - C(0)\}[1 + r(0)] - C(1)][1 + r(1)] - C(2) \\ &= \{L - C(0)\}[1 + r(0)][1 + r(1)] - C(1)[1 + r(1)] - C(2) \end{aligned}$$

In general we will have

$$V(i) = L \prod_{k=0}^{k=i-1} [1 + r(k)] - \sum_{j=0}^{j=i-1} C(j) \prod_{k=j}^{k=i-1} [1 + r(k)] - C(i)$$

At the end of the loan $V(mT) = 0$ so that with $i = mT$ we have

$$0 = L \prod_{k=0}^{k=mT-1} [1 + r(k)] - \sum_{j=0}^{j=mT-1} C(j) \prod_{k=j}^{k=mT-1} [1 + r(k)] - C(mT)$$

Dividing by $\prod_{k=0}^{k=mT-1} [1 + r(k)]$ we obtain

$$0 = L - \sum_{j=0}^{j=mT-1} \frac{C(j)}{\prod_{k=0}^{k=mT-1} [1 + r(k)]} \prod_{k=j}^{k=mT-1} [1 + r(k)] - \frac{C(mT)}{\prod_{k=0}^{k=mT-1} [1 + r(k)]}$$

or

$$0 = L - C(0) - \sum_{j=1}^{j=mT-1} \frac{C(j)}{\prod_{k=0}^{k=j-1} [1 + r(k)]} - \frac{C(mT)}{\prod_{k=0}^{k=mT-1} [1 + r(k)]}$$

The right hand side of the above equation is usually referred to as the *Net Present Value* of the loan. Each term is the present value of the respective cash flow. These values are also referred to as *discounted values*. We can write this as an *equation of value,* where the present value of the cash flows are equated to the net amount lent at time zero, as follows

$$\begin{aligned} L &= C(0) + \sum_{j=1}^{j=mT-1} \frac{C(j)}{\prod_{k=0}^{k=j-1} [1 + r(k)]} + \frac{C(mT)}{\prod_{k=0}^{k=mT-1} [1 + r(k)]} \\ &= C(0) + \sum_{j=1}^{j=mT} \frac{C(j)}{\prod_{k=0}^{k=j-1} [1 + r(k)]} \end{aligned}$$

Note that if we set $V(0) = 1 = L$ and all cash flows equal to zero then

$$V(i) = \prod_{k=0}^{k=i-1} [1 + r(k)]$$

which is the *accumulated value* of \$1 at the end of i periods. If $r(k) = r$ a constant for all time periods then

$$V(i) = [1 + r]^i$$

Example 4.3 *Consider a loan with payments at the end of each year of \$2000 for 3 years. Assume that the interest rate is 10% p.a. Calculate L for this loan.*

Solution 4.3 *We have $T = 3$, $m = 1$, $r(j) = 0.1$ for $j = 0, 1, 2$, $C(0) = 0$, $C(j) = 2000$ for $j = 1, 2, 3$ so that*

$$\begin{aligned} L &= \sum_{j=1}^{j=3} \frac{2000}{\prod_{k=0}^{k=j-1}[1.1]} \\ &= \frac{2000}{[1.1]} + \frac{2000}{[1.1]^2} + \frac{2000}{[1.1]^3} \\ &= 4973.70 \end{aligned}$$

Example 4.4 *Calculate the accumulated value of \$1 for 100 time periods at 5% per period.*

Solution 4.4 *We have the accumulated value at the end of 100 time periods given by*

$$V(100) = [1.05]^{100} = 131.50126$$

Exercise 4.2 *Consider a loan of \$10,000. Assume that payments of \$1,000 are made semi-annually in advance for 3 years. Assume also that the nominal annual interest rate on the loan is constant and equal to 10% p.a. Calculate $V(j)$ for $j = 0, 1, \ldots 6$ recursively. How much will be outstanding and payable at the end of the loan?*

If we work backwards from the maturity date of the loan, then we rearrange the difference equation as follows

$$V(j) = \frac{V(j+1) + C(j+1)}{[1 + r(j)]}$$

This is a *backward recursion* equation. Stepping back one time period at a time we get

$$\begin{aligned} V\left(mT-1\right) &= \frac{V\left(mT\right)+C(mT)}{\left[1+r\left(mT-1\right)\right]} \\ &= \frac{C(mT)}{\left[1+r\left(mT-1\right)\right]} \end{aligned}$$

for the last time period.

For the next to last period we have

$$\begin{aligned} V\left(mT-2\right) &= \frac{V\left(mT-1\right)+C(mT-1)}{\left[1+r\left(mT-2\right)\right]} \\ &= \frac{\frac{C(mT)}{\left[1+r(mT)\right]}+C(mT-1)}{\left[1+r\left(mT-2\right)\right]} \\ &= \frac{C(mT)}{\left[1+r\left(mT-2\right)\right]\left[1+r\left(mT-1\right)\right]}+\frac{C(mT-1)}{\left[1+r\left(mT-2\right)\right]} \end{aligned}$$

In general we have

$$V\left(mT-i\right)=\sum_{j=0}^{j=i-1}\frac{C(mT-j)}{\prod_{k=j+1}^{k=i}\left[1+r\left(mT-k\right)\right]} \quad \text{for } i=1 \text{ to } mT$$

Note that for $i=mT$ we have

$$V\left(0\right)=\sum_{j=0}^{j=mT-1}\frac{C(mT-j)}{\prod_{k=j+1}^{k=mT}\left[1+r\left(mT-k\right)\right]}$$

Example 4.5 *Write out in full the expression for* $V(5)$ *if* $T=8,\ \ m=1$ *for general* C *and* r*.*

Solution 4.5 *In this case* $mT-i=5$ *and* $mT=8$ *so that* $i=3,$ *therefore*

$$\begin{aligned} V\left(5\right) &= \sum_{j=0}^{j=2}\frac{C(mT-j)}{\prod_{k=0}^{k=2-j}\left[1+r\left(mT-k\right)\right]} \\ &= \frac{C(mT)}{\prod_{k=1}^{k=3}\left[1+r\left(mT-k\right)\right]}+\frac{C(mT-1)}{\prod_{k=2}^{k=3}\left[1+r\left(mT-k\right)\right]}+\frac{C(mT-2)}{\prod_{k=3}^{k=3}\left[1+r\left(mT-k\right)\right]} \end{aligned}$$

Noting that $V(0)=L-C(0)$ we then have

$$L-C\left(0\right)=\sum_{j=0}^{j=mT-1}\frac{C(mT-j)}{\prod_{k=j+1}^{k=mT}\left[1+r\left(mT-k\right)\right]}$$

or

$$L=C\left(0\right)+\sum_{j=1}^{j=mT}\frac{C(j)}{\prod_{k=0}^{k=j-1}\left[1+r\left(k\right)\right]}$$

4.2.5 Actuarial notation

Insurance premiums for long term insurance are usually paid as level premiums over the term of the insurance. Most loans are repaid with level repayments. It is also common to set the interest rate to be used for a loan or for calculating premiums for an insurance policy at the start of the loan or policy and to assume that the interest rate is constant for each period. It is also common in insurance policies for the premium to be payable in advance, whereas for loans it is normal to assume that payments are made in arrears. For these special cases where interest rates are constant and where payments are level there is an internationally accepted actuarial notation used by actuaries. We will introduce only some of this notation for some standard cases.

Assume that the interest rate is constant with $r(j) = i$ for $i = 0, 1, 2, \ldots, mT - 1$ and denote the number of time periods by $n = mT$. The cash flow is assumed to be level at 1 per period. If the level payments are assumed to be paid in arrears then $C(j) = 1$ for $j = 1, 2, \ldots, mT$ and $C(0) = 0$. Such a level stream of in arrears payments is called a *term certain annuity*. The value of this annuity is denoted by $a_{\overline{n}|}$ and is given by

$$\begin{aligned} a_{\overline{n}|} &= \sum_{j=1}^{j=n} \frac{1}{\prod_{k=1}^{k=j}[1+i]} \\ &= \sum_{j=1}^{j=n} \frac{1}{[1+i]^j} \\ &= \sum_{j=1}^{j=n} \left(\frac{1}{1+i}\right)^j \\ &= \frac{1}{[1+i]}\left[\frac{1-\left(\frac{1}{1+i}\right)^n}{1-\frac{1}{[1+i]}}\right] \\ &= \frac{1-\left(\frac{1}{1+i}\right)^n}{i} \end{aligned}$$

where we use the sum of a geometric progression to go from the third to the fourth line in order to derive the formula from the summation.

In actuarial notation we define

$$v = \frac{1}{1+i}$$

so that

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

Note that this is the value of an annuity payment of 1 per period in arrears for n periods at an interest rate of i per period. The periods could be months, quarters or years. As long as the period for the rate of interest is the same as the payment frequency this formula holds.

Example 4.6 *Calculate* $a_{\overline{10}|}$ *at rate of interest* $i = 0.01$ *per period.*

Solution 4.6 *Answer is* $\frac{1-v^n}{i} = \frac{1-\left(\frac{1}{1.01}\right)^{10}}{0.01} = : 9.4713$

Exercise 4.3 *Calculate the value of an annuity of \$1000 per month for 36 months at an annual nominal rate of interest of 6%p.a. with semi-annual compounding.*

In the case that payments are made in advance we have that $C(0) = 1$, $C(j) = 1$ for $j = 1, 2, \ldots, n-1$ and $C(n) = 0$. This is called an *annuity due*. In actuarial notation we place "double-dots" over the symbol to indicate that there are n payments made in advance in the annuity. The value is denoted by $\ddot{a}_{\overline{n}|}$ (pronounced "*a double dot n*") and we have

$$\begin{aligned}
\ddot{a}_{\overline{n}|} &= 1 + \sum_{j=1}^{j=n-1} \frac{1}{\prod_{k=1}^{k=j}[1+i]} \\
&= \sum_{j=0}^{j=n-1} \frac{1}{[1+i]^j} \\
&= \sum_{j=0}^{j=n-1} \left(\frac{1}{1+i}\right)^j \\
&= \left[\frac{1-\left(\frac{1}{1+i}\right)^n}{1-\frac{1}{[1+i]}}\right] \\
&= [1+i]\left[\frac{1-\left(\frac{1}{1+i}\right)^n}{i}\right] \\
&= [1+i]\, a_{\overline{n}|}
\end{aligned}$$

Example 4.7 *Calculate the value of a 5 year annuity due at 6% p.a.*

Solution 4.7 *Value* $= \ddot{a}_{\overline{5}|}$ *at 6%* $= [1.06]\left[\frac{1-\left(\frac{1}{1.06}\right)^5}{0.06}\right] = 4.4651$

Exercise 4.4 *Explain in words why you would expect* $\ddot{a}_{\overline{n}|} = [1+i]\, a_{\overline{n}|}$.

There are many other annuities with corresponding actuarial notation. We do not spend any more time on these here. The key tools that are needed to value any set of fixed cash flows have now been developed.

4.2.6 Leases and Zero-Coupon Bonds

In passing we mention a couple of financial instruments that are found in finance.

A specific type of loan used to finance the purchase of equipment is the lease. A *lease* is used where a piece of equipment is purchased by a financier and then leased, or rented, to the ultimate user of the equipment. The lease payments are usually made in advance and are level payments. A final payment at the end of the lease is also paid which is approximately equal to the value of the rented property at that time. This payment is called the *residual value.*

An important security is the zero-coupon bond. This security pays only one cash flow on maturity. The price of such a bond with a maturity payment at time T in the future of \$1 using an interest rate of $y(T)$ p.a. is denoted by $P(T)$ where

$$P(T) = \frac{1}{[1 + y(T)]^T}$$

T is the time to maturity and $y(T)$ is the *yield-to-maturity* for the zero coupon bond with a payment only at time T. Note that in actuarial notation this would be written as

$$P(T) = v^T$$

where $v = \frac{1}{1+y(T)}$.

In practice it is necessary to allow for the payment of taxation in the cash flows for an loan or a lease. It is the value of the *after-tax cash flows* that is the main interest in assessing loans and leases and cash flows in general. We do not cover the details of taxation here. It is important in practice to have an understanding of the taxation rules that apply and how these affect the cash flows that are received. These rules are often complex and vary from country to country.

Exercise 4.5 *Calculate the price of a zero coupon bond maturing in 5 years with a yield to maturity of 6.5% p.a. effective and a face value payable on maturity of \$10,000. Calculate the semi-annual compounding nominal interest rate equivalent to 6.5% p.a. Use this nominal rate to calculate the value of this zero coupon bond Comment on your answer.*

4.2.7 Continuous Compounding

There is an important link between survival models and compound interest. Consider the present value of a single payment of \$1 to be paid at time T. This is a zero-coupon bond. The value of this bond is denoted $P(T)$. As the maturity increases the present value of the bond decreases. Lets assume that as the maturity increases the **proportionate decrease** in the value of the bond is given by $\delta(\tau)$. Thus

$$\frac{-\frac{d}{d\tau}P(\tau)}{P(\tau)} = \delta(\tau)$$

so that

$$\frac{d\ln P(\tau)}{d\tau} = -\delta(\tau)$$

or

$$d\ln P(\tau) = -\delta(\tau)\,d\tau$$

Integrating from 0 to T we obtain

$$\int_0^T d\ln P(\tau) = \int_0^T -\delta(\tau)\,d\tau$$

Thus

$$\ln P(T) - \ln P(0) = \int_0^T -\delta(\tau)\,d\tau$$

But $P(0) = 1$ so that $\ln P(0) = 0$ and therefore

$$P(T) = \exp\left[\int_0^T -\delta(\tau)\,d\tau\right]$$

In actuarial terminology $\delta(\tau)$ is referred to as the *force of interest*.
To understand why this terminology is used, recall the survival model result

$$\frac{s(x+t)}{s(x)} =_t p_x = \exp\left[\int_x^{x+t} -\mu(y)\,dy\right]$$

If we replace x with 0, $x+t$ with T and y with τ then we have

$$_Tp_0 = \exp\left[\int_0^T -\mu(\tau)\,d\tau\right]$$

and we immediately see that the force of interest plays a similar role in finance as the force of mortality plays in survival models.

In the finance literature the term $\delta(\tau)$ is referred to as the *instantaneous forward rate* of interest at time 0 applying over time τ to $\tau + d\tau$.

We note also that if we assume that $\delta(\tau) = \delta$ a constant then

$$P(T) = \exp[-\delta T]$$

Recall also that for a constant interest rate per period of r

$$\begin{aligned} P(T) &= \frac{1}{[1+r]^T} \\ &= [1+r]^{-T} \end{aligned}$$

Therefore

$$\exp[-\delta T] = [1+r]^{-T}$$

or

$$-\delta T = -T \ln[1+r]$$

or

$$\delta = \ln[1+r]$$

so that the continuous compounding interest rate is the natural log of 1 plus the discrete per period rate of interest.

Example 4.8 *Calculate the continuous compounding p.a. equivalent interest rate to an annual effective rate of 10% p.a.*

Solution 4.8 *We have* $\delta = \ln[1+r] = \ln[1.1] = 0.09531$.

4.3 Life Annuities

Governments raise funding for a host of reasons. In modern times the government borrows money to finance roads, hospitals, education and other services as well as meeting the costs of social security and government pension benefits. In past times, governments often raised funds to finance wars. In the 1600's various governments used annuities to borrow funds from individuals where the repayments would be made as long as the lender was alive. Thus the government was borrowing money by issuing *life annuities*.

A life annuity is an annuity of level payments made during the life of an individual. As long as the life is alive then a regular annuity payment will be made. As soon as the life dies then the annuity payments cease. These annuities are *contingent on human life*. Individuals who bought life annuities from the government were taking a gamble on their life in the sense that if they lived longer than expected then they would receive more payments than a life who died earlier than expected.

Note that the level premium payments on a life insurance policy are only paid as long as the life is alive and cease on the life's death. Thus level premium payments for a life insurance policy are a life annuity. A pension paid to an individual after retirement as long as they are alive is also a life annuity.

Assume that a life annuity is issued for an amount A, often called *the consideration*, to a life aged x (in periods) with level annuity payments payable in advance of P per period. Assume also that the oldest age possible in periods is ω. The interest rate for the annuity is assumed to be a constant of i per period. Let the expected value of the annuity at the beginning of time period j ($j = 1, 2, \ldots, \omega - x$) immediately **before** any annuity due at time j has occurred be denoted by $V(x + j)$. This is an expected value since, unlike in the case of a fixed annuity, we do not know the actual cash flows that will occur since the number of cash flows is a random variable depending on the survival of the life. We can however calculate an expected value of the future cash flows.

Note that time period j covers the period from time $j - 1$ to j. If the life dies during time period j aged $x + j - 1$ periods then no further payments are made with the last payment made at the start of period j. If the life survives to the end of the time period then a payment of P is made at the start of the period and further payments are made as long as the life is alive. The payment at the start of the next period is included in $V(x + j)$.

Assume that we have a survival model for a life aged x with $\frac{s(x+j)}{s(x)}$ for $j = 1, 2, \ldots, \omega - x$. We have $p_{x+j} = \frac{s(x+j+1)}{s(x+j)}$ for $j = 0, 1, 2, \ldots, \omega - x - 1$ and $q_{x+j} = 1 - p_{x+j}$. Consider the expected value of the life annuity for the life when she is aged $x + j$ immediately before payment of the annuity then due denoted by $V(x + j)$. If the life dies during the next period, with probability q_{x+j}, then the expected value of the life annuity at the end of the period will be 0 since no further payments will be made. However if the life survives the next period, with probability p_{x+j}, then the expected value of the life annuity at the end of the period will be equal to the expected value of a life annuity for a life aged $x + j + 1$ (i.e. $V(x + j + 1)$). In both cases an annuity payment of P is received at the start of the period. The expected value at the end of the period will therefore be

$$\begin{aligned} & q_{x+j}\{0\} + p_{x+j}\{V(x+j+1)\} \\ = \; & p_{x+j}\{V(x+j+1)\} \end{aligned}$$

Now, in order to equate this to the expected vale at the beginning of the time period we need to allow for interest and the payment of P made at the start of the period. In the case of a loan we noted that the value of the loan at the start of the period was increased with interest over the period. The same applies in the case of the expected value of the life annuity. The expected value and the payment at the start of the period also be increased by interest from the beginning to the end of each time period. Thus we have

$$\{V(x+j)\}[1+i] = \{P\}[1+i] + p_{x+j}\{V(x+j+1)\}$$

or

$$V(x+j) = P + \frac{p_{x+j}\{V(x+j+1)\}}{[1+i]}$$

At the end of the longest time period that it is assumed possible to live we have $V(\omega) = 0$ with the final payment at $\omega - 1$. Working backwards from the ultimate age we have

$$V(\omega - 1) = P$$

$$\begin{aligned} V(\omega - 2) &= P + \frac{p_{\omega-2}\{V(\omega-1)\}}{[1+i]} \\ &= P + \frac{p_{\omega-2}\{P\}}{[1+i]} \end{aligned}$$

$$\begin{aligned} V(\omega - 3) &= P + \frac{p_{\omega-3}\{V(\omega-2)\}}{[1+i]} \\ &= P + \frac{p_{\omega-3}\left\{P + \frac{p_{\omega-2}\{P\}}{[1+i]}\right\}}{[1+i]} \\ &= P + \frac{p_{\omega-3}\{P\}}{[1+i]} + \left\{\frac{p_{\omega-3}p_{\omega-2}\{P\}}{[1+i]^2}\right\} \end{aligned}$$

and in general

$$V(\omega - j) = P + \sum_{k=1}^{k=j-1} \frac{\prod_{l=0}^{l=k-1} p_{\omega-j+l}\{P\}}{[1+i]^k}$$

Noting that

$$\begin{aligned} &\prod_{l=0}^{l=k-1} p_{\omega-j+l} \\ = \; &\frac{s(\omega-j+1)}{s(\omega-j)} \frac{s(\omega-j+2)}{s(\omega-j+1)} \cdots \frac{s(\omega-j+k)}{s(\omega-j+k-1)} \\ = \; &\frac{s(\omega-j+k)}{s(\omega-j)} \\ = \; &{}_k p_{\omega-j} \end{aligned}$$

we can write

$$\begin{aligned} V(\omega - j) &= P + \sum_{k=1}^{k=j-1} \frac{{}_k p_{\omega-j}\{P\}}{[1+i]^k} \\ &= \sum_{k=0}^{k=j-1} \frac{{}_k p_{\omega-j}\{P\}}{[1+i]^k} \end{aligned}$$

If we now let $\omega - j = x$ then we obtain the expected value of the life annuity at age x as

$$V(x) = \sum_{k=0}^{k=\omega-x-1} \frac{{}_kp_x P}{[1+i]^k}$$

Note that the life annuity to a life currently aged x only pays an amount of P provided the life is alive at age $x+k$ and pays an amount of zero if the life is dead. The probability that the life is alive at age $x+k$ is $\frac{s(x+k)}{s(x)} =_k p_x$. Thus the expected payment for a life currently aged x at age $x+k$ will be

$${}_kp_x \{P\} + (1 -_k p_x)\{0\} =_k p_x P$$

If we discount this for interest then its value at age x will be

$$\frac{{}_kp_x P}{[1+i]^k}$$

and if we sum over all ages then we get the formula derived above. In this sense we can see that the expected present value of the annuity payments is a "discounted expected value".

Some life annuities are only payable for a fixed time period or until the death of the life whichever occurs first. Thus if payments will only be paid for n time periods or until death of the life whichever occurs first it will be necessary to modify the valuation formula to exclude any payments made when the life lives longer than n time periods. There are many other variations that can be found in practice including increasing life annuities, indexed life annuities, and decreasing life annuities. The details of these cases are left for later study for those who wish to complete further study in actuarial science.

4.3.1 Actuarial Notation

The (expected) value of a life annuity due of 1 per period paid to a life aged x as long as they are alive is denoted by the actuarial notation $\ddot{a}_x$. From the above we have

$$\ddot{a}_x = \sum_{k=0}^{k=\omega-x-1} \frac{{}_kp_x}{[1+i]^k}$$

Recall that we have

$${}_kp_x = \exp\left[\int_x^{x+k} -\mu(\tau)\, d\tau\right]$$

which is the expected value of the payment of \$1 at time $x+k$. From earlier, we can also write

$$\frac{1}{[1+i]^k} = \exp\left[\int_x^{x+k} -\delta(\tau)\, d\tau\right]$$

where this is a discount factor to present value the expected payment.

Substituting into the formula for $\ddot{a}_x$ we obtain

$$\begin{aligned}\ddot{a}_x &= \sum_{k=0}^{k=\omega-x-1} \exp\left[\int_x^{x+k} -\mu(\tau)\,d\tau\right] \exp\left[\int_x^{x+k} -\delta(\tau)\,d\tau\right] \\ &= \sum_{k=0}^{k=\omega-x-1} \exp\left[\int_x^{x+k} -\left[\mu(\tau)+\delta(\tau)\right] d\tau\right]\end{aligned}$$

Note that this can be interpreted as a "discounted expected value" of the annuity payments.

Exercise 4.6 *Assume that* $\mu(x) = \frac{1}{\omega - x} \quad 0 \leq x < \omega$ *and that the force of interest is a constant equal to* δ*. Determine an expression for a life annuity of 1 p.a. to a life aged* x *on these assumptions.*

Exercise 4.7 *If* a_x *is the value of a life annuity with payments in arrears, show that* $\ddot{a}_x = 1 + a_x$

Exercise 4.8 *Use the IA64-70 Ultimate life table and an interest rate of 5% p.a. to determine the expected value of a life annuity due of 1000 p.a. for a life aged 50. (Hint - use a spreadsheet).*

4.4 Investment Management

Insurance companies invest the premiums that they receive from the sale of insurance policies. They invest these funds until they are required to pay claims or are declared as surplus or profits and paid out as dividends to shareholders and/or policyholders. Superannuation funds invest member and employer contributions until they are required to pay out benefits on retirement, death or withdrawal of the members of the funds. Some insurance companies invest the funds directly but many invest through separate *investment or fund managers*.

Fund managers are companies that specialise in investing funds in various markets. Most superannuation funds are invested through fund managers. These fund managers are given investment objectives to meet and their investment performance is monitored and compared with the objectives. Asset consultants specialise in assisting insurance companies and superannuation funds set investment objectives, select fund managers and monitor the investment return performance of the managers.

Actuaries are often involved in recommending investment policies or strategies to insurance company management and trustees of superannuation funds and also in providing advice on investment manager selection. They may do this as part of their role in an insurance company or they may work for an asset consulting firm.

Insurance and superannuation funds can invest directly in assets or through unit trusts or other pooled funds. Direct investment is usually appropriate for larger funds whereas pooled arrangements are often more appropriate for smaller funds.

In this section we briefly introduce the terminology used in investment management.

4.4.1 Asset classes

Insurance and superannuation funds invest in a range of asset classes. Each of these asset classes have common features. Within each asset class individual securities are selected by the fund managers based on an analysis of each individual security. The process of assessing individual securities is referred to as *security analysis.* We will not discuss security analysis and will mainly outline issues in making investment decisions at the asset class level. Studies suggest that most of the return from an investment strategy comes from selecting the percentage of funds placed in each asset class rather than the particular assets held in the asset class. This is partly because the amount of money invested by insurance and superannuation funds is large and they normally hold a *well diversified* portfolio of securities in each of the asset classes.

Equities

Equities represent ownership interests in companies. They are the *residual claimants* of the net assets of the company after other creditors have been satisfied in the event of bankruptcy. Equities are held in the form of *shares* in the company. They receive dividends on a regular basis, usually semi-annually in Australia. Dividends are cash payments made from the profits of the company. The dividends are not guaranteed and depend on the company being sufficiently profitable. The value of the shares is also not guaranteed and is determined by the price that can be obtained by selling the shares in the market. Dividends often have imputation credits in Australia. These credits are an allowance against an individuals tax for tax payments made already by the company as company tax.

Fixed Interest

Fixed interest or debt securities pay fixed cash flows, referred to as *interest coupons*, on a regular basis and a maturity value on a future specified date. Governments issue fixed interest securities as do companies. In the case of a government security, it is unlikely that the interest and principal will not be paid although it is possible that the government may suspend payments as happened recently in Russia.

Company or private fixed interest securities have a chance that the interest or principal will not be paid. This is called *default.* In the event of a default, the fixed interest security holders usually have the right to appoint managers of the company who take over from the existing management and whose job is to ensure that the debt holders receive repayment if possible.

Inflation indexed

Some securities are inflation indexed. These include government indexed bonds and private sector inflation indexed annuities and bonds. The principal is usually indexed to an inflation index such as the CPI. Interest payments are usually made as a fixed percentage of the indexed principal amount. Securities with the principal indexed for inflation are called capital indexed bonds.

Cash

Short term investments in bank accounts and interest bearing securities of less than a year to maturity are usually treated as cash. This includes bank bills, promissory notes and Treasury notes. These are securities that are issued at a discount to their face value.

Property

Fixed property in the form office buildings, shopping centres and factories are purchased by institutional investors for investment purposes. These properties provide a return in the form of rentals and capital growth. The rental income is determined under leases to tenants. The leases usually provide for rent reviews on a regular basis. Often the rent increase with inflation.

Derivatives

Derivatives are securities that derive their value or cash flows from the value of other securities. It is possible to use derivatives to gain investment exposure to asset classes without directly investing in the physical securities. The basic derivative securities include *options, futures, and swaps.* Options allow the holder the option to buy or sell a security on a future date. Futures are agreements to buy or sell on a future date with no option. Swaps involve the exchange of cash flows on one asset for the cash flows on another asset.

There are many *exotic derivatives* available as well. An important feature of derivatives is that they usually involve *leverage*. Leverage involves borrowing of cash in order to increase the exposure to an asset class to a level higher than your own capital.

4.4.2 Asset Allocation

The key decision that a fund has to make is how to allocate the funds derived from premiums or contributions into the different asset classes and how to change that allocation through time. Most funds will set a long term asset allocation considered appropriate for their business and then depart from this long term asset allocation according to market circumstances. The long run allocation specifies the percentage of the total funds to be invested in each asset class. Thus it could be specified as

30% in domestic equities, 30% in international equities, 20% in domestic bonds, 10% in international bonds and 10% in cash.

Strategic asset allocation

The strategic asset allocation is the long run asset allocation appropriate for an insurance company or a superannuation fund which is usually set based on the liabilities that the fund will have to meet. Thus a general insurance company with claims expected to be payable over the next 3 to 5 years in respect of the current business, and with these expected claims likely to increase with inflation, will usually invest more in fixed interest and inflation indexed securities and less in equities and property. A superannuation fund with young members who are looking for the fund investments to provide adequate pensions in retirement is more likely to have significant holdings in equities and property, also referred to as *growth assets*.

Tactical asset allocation

Once a strategic asset allocation has been established then the fund manager is required to invest to meet the long run objectives of the fund. However the fund managers are often required to maximize the returns from the funds invested subject to meeting the strategic objectives over the longer term. In order to do this the fund managers often use *tactical asset allocation*. This is where a fund manager varies the percentage of the funds in various asset classes in order to maximize the return from the funds invested. For example, if the manager considers that the returns in equities will be lower then they will underweight equities compared with the other asset classes. This departure from the long run strategic asset allocation to take advantage of expected differences in relative returns is referred to as tactical asset allocation.

4.4.3 Types of Fund Manager

Different fund managers use different approaches to invest the funds that they manage.

Balanced managers

A *balanced manager* manages all of the asset classes for a fund and varies the percentages in the different asset classes in order to maximize the returns subject to constraints imposed by the long run strategic asset allocation. Thus they set asset allocations for all asset classes and manage the funds in the different asset classes.

Specialist managers

Specialist managers invest in particular asset classes and do not manage the investments across different asset classes. Thus there will be managers who specialise in domestic equities, fixed interest and so on. A superannuation fund may select different specialist managers for each of the main asset classes. They will then monitor the total performance. They may use an asset consultant to set the strategic asset allocation which determines the amount given to each specialist manager.

Indexed funds

There is an increasing use of index funds for each of the asset classes. The managers who offer indexed fund invest in an asset class in the same proportions as for a market index such as the All-Ordinaries index for Australian shares. The cost of managing an index fund is much less than for specialist managers.

Exercise 4.9 *Obtain the prospectus for a retail managed fund available in the Australian market. Read the prospectus and comment on the main features used to promote the fund and the size of the fees charged.*

4.5 Conclusions

This chapter has introduced the financial and insurance mathematics required to value cash flows. We used recurrence relations to demonstrate how the funds evolve through time and we then used these recurrence relations to derive present value formulae. We showed how we can derive expected values of cash flows that are contingent on survival. We also introduced the basic actuarial notation for annuities including an annuity-certain and a life annuity. Finally we briefly introduced the terminology of funds management.

4.6 Solutions to Exercises

Ex 4.1 *Require $j^{(4)}$ the nominal rate with quarterly compounding corresponding to $i = 0.075$.*

$$\left(1 + \frac{j^{(4)}}{4}\right)^4 = 1.075$$

or

$$\begin{aligned} j^{(4)} &= 4\left[(1.075)^{\frac{1}{4}} - 1\right] \\ &= 0.072978 \end{aligned}$$

Thus the rate is 7.3% p.a.
In Excel the function
NOMINAL(0.075,4)
will calculate the nominal rate with quarterly compounding equivalent to 7.5% per annum effective.

Ex 4.2 *Payments are semi-annual so over three years we will have 6 payments. We just apply the recursive formula with* $r(j) = \frac{0.1}{2} = 0.05$*,* $C(j) = 1,000$ *for* $j = 0$ *to* 5*. Note that in the text* $V(j)$ *is defined as the value after the payment due at time* j *has been paid.*
We then have that $V(0) = 10,000 - 1000 = 9,000$*,* $V(1) = 9,000(1.05) - 1,000 = 8,450$ *etc. The values obtained are given below:*

Nominal rate	10.00%	
Effective semi-annual rate	5.00%	
Loan	10000.00	
j	**V(j)**	**C(j)**
0	9000.00	1000.00
1	8450.00	1000.00
2	7872.50	1000.00
3	7266.13	1000.00
4	6629.43	1000.00
5	5960.90	1000.00
6	6258.95	

The amount outstanding at the end of the loan in three years (six half years) will be $6,258.95.
Note that we can use the Excel function
FV(0.05,j,1000,-10000,1)
to calculate the future value of this stream of cash flows. This gives the values just before the payment then due at time j. Thus to get the $V(j)$ *values above you would need to calculate*
FV(0.05,j,1000,-10000,1)-1000.

Ex 4.3 *The interest rate is nominal with semi-annual compounding assumed. But we have to value monthly cash flows of an annuity so we need a monthly per period effective rate. Thus we must convert the interest rate into a monthly rate. The annual effective interest rate equivalent to 6% p.a. with semi-annual compounding is given by*

$$\left(1 + \frac{0.06}{2}\right)^2 = 1 + i$$

so that $i = 0.0609$*. This can be determined using the Excel function EFFECT(0.06,2).*
The nominal annual rate with monthly compounding equivalent to this is given by

$$\left(1 + \frac{j^{(12)}}{12}\right)^{12} = 1.0609$$

which gives

$$\begin{aligned} j^{(12)} &= 12\left[(1.0609)^{\frac{1}{12}} - 1\right] \\ &= 0.05926 \end{aligned}$$

*or 5.926% p.a. This can be determined using the Excel function NOMINAL(*0.0609*,12).*
In fact the two calculations above can be done in one step using NOMINAL(EFFECT(0.06,2),12)
The monthly effective rate per period is therefore $\frac{0.05926}{12} = 0.00494$.
The annuity is an ordinary annuity (not an annuity due) so payments are in arrears. The value of the annuity is

$$
\begin{aligned}
1000a_{\overline{36}|} \text{ at rate } i &= 0.00494 \\
&= 1000\frac{1-\left(\frac{1}{1+i}\right)^{36}}{i} \\
&= 1000\frac{1-0.837484}{0.00494} \\
&= 1000 \times 32.907103 \\
&= 32,907.10
\end{aligned}
$$

In Excel we can calculate this value using the PV function with PV(NOMINAL(EFFECT(0.06,2),12)/12,36,-1000,0,0).

Ex 4.4 *The value of n payments in advance is given by* $\ddot{a}_{\overline{n}|}$. *If we consider the value of n payments in arrears this is given by* $a_{\overline{n}|}$. *In the former case all payments are received one period before the payments are received in the latter case. Thus each $1 payment in the former case can be invested for one period to generate payments in arrears. However the amount of the payment in arrears will be* $(1+i)$ *including the interest. Thus the value of* $\ddot{a}_{\overline{n}|}$ *will be the value of n payments of* $(1+i)$ *in arrears which is* $(1+i)\,a_{\overline{n}|}$.

Ex 4.5 *The price of a $10,000 zero coupon bond maturing in 5 years with yield to maturity 6.5% p.a. is*

$$
\begin{aligned}
10,000P(0,5) &= 10,000\left(\frac{1}{1+0.065}\right)^5 \\
&= 7,298.81
\end{aligned}
$$

This can be calculated in Excel using PV(0.065,5,0,-10000,0).
Note that in the PV function we set the payment to 0 indicating that it is a zero coupon bond.
The semi-annual nominal compounding rate is given by

$$\left(1+\frac{j^{(2)}}{2}\right)^2 = 1.065$$

so that $j^{(2)} = 0.063977$. *In excel we use NOMINAL(0.065,2).*
The value of the zero coupon bond using the nominal rate will be

$$\begin{aligned} & 10,000\left(\frac{1}{1+\frac{0.063977}{2}}\right)^{10} \\ = \; & 7,298.81 \end{aligned}$$

This can be calculated in Excel using
PV(NOMINAL(0.065,2)/2,10,0,-10000,0).
Both answers are the same as would be expected since we have used an equivalent interest rate. As long as we adjust the number of periods to be consistent with the compounding frequency then we will obtain consistent answers.

Ex 4.6 *The value of a life annuity of 1 p.a. is*

$$\begin{aligned} a_x &= \sum_{k=0}^{k=\omega-x-1} ({}_kp_x)\left(\frac{1}{1+i}\right)^k \\ & \sum_{k=1}^{k=\omega-x-1} \exp\left[\int_x^{x+k} -\mu(\tau)\,d\tau\right]\exp\left[\int_x^{x+k} -\delta d\tau\right] \\ &= \sum_{k=1}^{k=\omega-x-1} \exp\left[\int_x^{x+k} -\left[\frac{1}{\omega-\tau}+\delta\right]d\tau\right] \\ &= \sum_{k=1}^{k=\omega-x-1} \exp\left[\left[\ln(\omega-\tau)+\delta\tau\right]_x^{x+k}\right] \\ &= \sum_{k=1}^{k=\omega-x-1} \frac{\omega-(x+k)}{\omega-x}\exp\left[-\delta k\right] \end{aligned}$$

(Note: we could have done this for a life annuity due with payments in advance as well).

Ex 4.7 *We have that*

$$\begin{aligned} \ddot{a}_x &= \sum_{k=0}^{k=\omega-x-1} \frac{{}_kp_x}{[1+i]^k} \\ &= 1+\sum_{k=1}^{k=\omega-x-1} \frac{{}_kp_x}{[1+i]^k} \\ &= 1+a_x \end{aligned}$$

or by general reasoning we can see that a life annuity due has a payment immediately and then it is the same as an life annuity with payments in arrears.

Ex 4.8 *Require*

$$1,000\ddot{a}_{50} = \sum_{k=0}^{k=\omega-49} \frac{{}_kp_{50}}{[1+i]^k}$$

Using IA64-70 and 5% interest we get $14344.05. Calculations are set out over the page.

Ex 4.9 *You should consider how many different retail funds are available, what are their features, what exactly are the fees charged such as management expense ratios, fees, commissions.*

IA64-70 Ultimate	Interest rate	0.05			
Age	**qx**	**lx**	**kpx**	**(1+I)^-k**	PV
50	0.00489	999,999	1.00000	1.00000	1.00000
51	0.00547	995109.0	0.99511	0.95238	0.94772
52	0.00612	989665.8	0.98967	0.90703	0.89766
53	0.00685	983609.0	0.98361	0.86384	0.84968
54	0.00766	976871.3	0.97687	0.82270	0.80368
55	0.00856	969388.4	0.96939	0.78353	0.75954
56	0.00956	961090.5	0.96109	0.74622	0.71718
57	0.01067	951902.5	0.95190	0.71068	0.67650
58	0.01190	941745.7	0.94175	0.67684	0.63741
59	0.01327	930538.9	0.93054	0.64461	0.59983
60	0.01477	918190.6	0.91819	0.61391	0.56369
61	0.01643	904629.0	0.90463	0.58468	0.52892
62	0.01826	889765.9	0.88977	0.55684	0.49546
63	0.02027	873518.8	0.87352	0.53032	0.46325
64	0.02249	855812.6	0.85581	0.50507	0.43224
65	0.02491	836565.3	0.83657	0.48102	0.40240
66	0.02758	815726.5	0.81573	0.45811	0.37369
67	0.03049	793228.8	0.79323	0.43630	0.34608
68	0.03368	769043.2	0.76904	0.41552	0.31955
69	0.03716	743141.8	0.74314	0.39573	0.29409
70	0.04096	715526.7	0.71553	0.37689	0.26967
71	0.04511	686218.7	0.68622	0.35894	0.24631
72	0.04962	655263.4	0.65526	0.34185	0.22400
73	0.05452	622749.2	0.62275	0.32557	0.20275
74	0.05984	588796.9	0.58880	0.31007	0.18257
75	0.06561	553563.3	0.55356	0.29530	0.16347
76	0.07186	517244.0	0.51724	0.28124	0.14547
77	0.07861	480074.9	0.48008	0.26785	0.12859
78	0.08590	442336.2	0.44234	0.25509	0.11284
79	0.09376	404339.5	0.40434	0.24295	0.09823
80	0.10221	366428.6	0.36643	0.23138	0.08478
81	0.11129	328976.0	0.32898	0.22036	0.07249
82	0.12102	292364.2	0.29236	0.20987	0.06136
83	0.13144	256982.3	0.25698	0.19987	0.05136
84	0.14255	223204.6	0.22320	0.19035	0.04249
85	0.15440	191386.7	0.19139	0.18129	0.03470
86	0.16698	161836.6	0.16184	0.17266	0.02794
87	0.18033	134813.2	0.13481	0.16444	0.02217
88	0.19445	110502.3	0.11050	0.15661	0.01731
89	0.20934	89015.1	0.08902	0.14915	0.01328
90	0.22501	70380.7	0.07038	0.14205	0.01000
91	0.24145	54544.3	0.05454	0.13528	0.00738
92	0.25863	41374.6	0.04137	0.12884	0.00533
93	0.27655	30673.9	0.03067	0.12270	0.00376
94	0.29517	22191.0	0.02219	0.11686	0.00259
95	0.31445	15640.9	0.01564	0.11130	0.00174
96	0.33435	10722.6	0.01072	0.10600	0.00114
97	0.35481	7137.5	0.00714	0.10095	0.00072
98	0.37578	4605.1	0.00461	0.09614	0.00044
99	0.39719	2874.6	0.00287	0.09156	0.00026
100	0.41896	1732.8	0.00173	0.08720	0.00015
101	0.44101	1006.8	0.00101	0.08305	0.00008
102	0.46326	562.8	0.00056	0.07910	0.00004
103	0.48563	302.1	0.00030	0.07533	0.00002
104	0.50802	155.4	0.00016	0.07174	0.00001
105	0.53036	76.4	0.00008	0.06833	0.00001
106	0.55254	35.9	0.00004	0.06507	0.00000
107	0.57449	16.1	0.00002	0.06197	0.00000
108	0.59613	6.8	0.00001	0.05902	0.00000
109	0.61738	2.8	0.00000	0.05621	0.00000
110	0.63817	1.1	0.00000	0.05354	0.00000
111	1.00000	0.4	0.00000	0.05099	0.00000

Chapter 5
ECONOMICS OF RISK

5.1 Learning Objectives

The main objectives of this chapter are:

- to introduce the concept of expected utility and to show how it can be used to determine preferences over different risks,
- to introduce the concept of time preference and show how it relates to the market rate of interest for valuing cash flows, and
- to show how expected utility can be used to value risky cash flows.

5.2 Risk and Expected Utility

5.2.1 Introduction

A major focus of actuarial science is the quantification of risk. Actuaries are involved in the design of products that manage risk in various ways. The early life insurance products used the concepts of pooling and profit sharing to manage the risk of death. Risk pooling and sharing of the costs of risk are major functions of insurance and retirement products.

In order to quantify risk we need to understand some basic principles of insurance and financial economics. Micro-economics is of most interest to actuaries since this deals with the theory required to understand the economics of insurance and finance.

Risk can be defined in many different ways. For our purposes we will define *risk* as future uncertainty. Any situation where the outcome is uncertain will be considered risky. Risk is sometimes only considered as occurring where adverse outcomes are possible. There is also distinction drawn between *speculative risk* and *pure risk*. Pure risk is usually defined as exposure to adverse outcomes where insurance would normally be used. Speculative risk being defined as exposure to both favorable and adverse outcomes. This distinction is artificial since future uncertainty involves both favorable and adverse outcomes and we should consider risk in totality. Such an approach to risk is sometimes referred to as *holistic risk management*.

A major reason for quantifying risk is to assist in deciding on whether or not to assume certain risks or to insure against them. We also need some means of ranking or ordering different risks so that we can decide which risks we prefer.

5.2.2 Expected values

One approach to quantifying uncertainty would be to use the expected value of an uncertain outcome. Thus for a particular risk we would develop a probability model where the random variable(s) of interest was determined for all of the possible outcomes and positive probabilities assigned to each possible outcome. The probability density for these random variable(s) would be used to evaluate expected values. This probability density would usually be based on the *actual* or *real world probabilities* but it could also be based on subjective probabilities based on future expectations. This expected value would then be used to rank or order different risks. The higher the expected value the more preferred the risk.

The expected value of a risk is sometimes referred to as the *actuarial value of a risk*. The use of this terminology is misleading since actuaries do not normally use the expected value to assess risk. The usual practice is for actuaries to modify the probabilities of future events before calculating the expected value so that the value includes a safety margin. The probabilities used are selected so as to produce *conservative* values. Thus in life insurance the mortality rates or survival probabilities are sometimes adjusted by a percentage (or proportional hazard) before calculating an expected value. This results in an expected value that is different to that resulting from using the actual survival probabilities.

Example 5.1 *In calculating the premium for a term life insurance product an allowance for risk can be made by increasing the mortality rates used to calculate the expected value of the loss. Discuss what happens to the premium in this case.*

Solution 5.1 *By increasing the mortality rates the survival probabilities are lower. This means that the expected survival times are lower and the sum insured will be assumed payable earlier than for the actual probabilities. This results in a higher premium since the average payment time is earlier and the present value is higher. The increased premium includes a risk loading for the mortality risk.*

Exercise 5.1 *How would you adjust the mortality rates to include a risk loading when calculating the value for a life annuity product? Comment on the difference between this case and the term life insurance case.*

Recall that the expected value of a discrete random variable X is just the probability weighted values of the random variable. Thus

$$E[X] = \sum_x \Pr[X = x] \cdot x$$

The expected value is not the best way to compare or evaluate risks. This is because it is also necessary to consider the range of possible outcomes, not just their expected values. If we had two risks, both with the same expected value and one that had much more adverse extreme outcomes than the other, then we would probably prefer the risk with the less extreme outcomes. One measure of the range of possible outcomes is given by the variance of the random variable. This is the second moment about the expected value. To calculate a variance, we take the probability weighted sum of the squares of the difference between the value of the random variable and the expected value. Thus

$$var\left[X\right] = \sum_{x} \Pr\left[X = x\right].\left(x - E\left[X\right]\right)^2$$

The *standard deviation* is the square root of the variance. It has the same units as the mean and is often more meaningful for comparison purposes than the variance.

Consider the following example.

Example 5.2 *You have a choice of investing $10,000 in two potential investments. You must invest in one or the other. The pay-offs from the investments are as follows*

Outcome	*Probability*	*Investment A*	*Investment B*
Good	$\frac{1}{10}$	*50,000*	*26,000*
Middle	$\frac{22}{25}$	*12,500*	*15,000*
Bad	$\frac{1}{50}$	*0*	*10,000*

Which investment would you select?

Solution 5.2 *The expected values of the investments are*

$$E\left[A\right] = \frac{1}{10}50,000 + \frac{22}{25}12,500 + \frac{1}{50}0 = 16,000$$

and

$$E\left[B\right] = \frac{1}{10}26,000 + \frac{22}{25}15,000 + \frac{1}{50}10,000 = 16,000$$

Thus both have the same expected values.
Few people would select Investment A.
We can calculate the variance of both of these investments as follows

$$\begin{aligned} var\left[A\right] &= \frac{1}{10}\left(50,000 - 16,000\right)^2 + \frac{22}{25}\left(12,500 - 16,000\right)^2 + \frac{1}{50}\left(0 - 16,000\right)^2 \\ &= 131,500,000 \end{aligned}$$

(standard deviation of 11,467.34) and

$$\begin{aligned} var\,[B] &= \frac{1}{10}(26,000-16,000)^2+\frac{22}{25}(15,000-16,000)^2+\frac{1}{50}(10,000-16,000)^2 \\ &= 11,600,000 \end{aligned}$$

(standard deviation of 3,405.88).
Thus investment B has a much lower variability around the expected value. Many investors regard an investment with the same expected value but lower variability than another investment as less risky.

Sometimes the financial consequences of a risk can influence decisions about risk. Consider the following examples.

Example 5.3 *A fair die is rolled. You receive \$100,000 if a 1 appears face up, otherwise you receive zero. What is the expected value of the pay-off from this game? Would you pay this amount to partake in this game? Assume that your total wealth is \$100,000.*

Solution 5.3 *The probability of a 1 is* $\frac{1}{6}$. *Thus the expected value of the pay-off is* $\frac{100000}{6} = 16,666.67$. *If you pay* \$16,666.67 *to play then you could lose this amount with probability* $\frac{5}{6}$. *The outcomes of participating in this game, assuming you pay the expected value to participate, are a total wealth of \$200,000 with probability* $\frac{1}{6}$ *(if a 1 appears) and a total wealth of \$83,333.33 with probability* $\frac{5}{6}$ *(if a 1 does not appear). If you do not participate in the game then you will have a wealth of \$100,000 with probability* $\frac{1}{6}$ *(if a 1 appears) and \$100,000 with probability* $\frac{5}{6}$ *(if a 1 does not appear). In order to decide if you will participate in the game you need to express a preference for one of these two wealth distributions i.e. we need to rank the alternatives. Many individuals would not be happy to lose this much on a gambling game even if there was a chance of winning \$100,000 and even if the game was fair. However such a choice depends on an individual's preferences.*

Example 5.4 *Say that there is a* $\frac{1}{6}$ *chance that you will lose \$100,000 and that you can buy insurance against this loss for \$16,666.67. Your total wealth is \$100,000. Would you buy the insurance?*

Solution 5.4 *In deciding to buy the insurance you are expressing a preference over two wealth distributions. If you do not buy the insurance then your total wealth would be 0 with probability* $\frac{1}{6}$ *or \$100,000 with probability* $\frac{5}{6}$. *If you buy the insurance then your total wealth will be \$83,333.33 with certainty (regardless of whether you incur the loss or not). Most individuals faced with such a risk would be prepared to pay this much to avoid such a large loss with such a high probability. The consequences otherwise would be a total loss of wealth. Many individuals would be willing to pay more than the expected value of the loss for the insurance.*

In the first example the individual can exchange a certain amount of money in return for an uncertain gamble with an expected value equal to the cost of entering the gamble. In the second example the individual can exchange an uncertain outcome in the form of a loss for a certain payment equal to the expected value of the loss. In both cases it is necessary to express a preference over future wealth distributions for different outcomes.

The possible outcomes are usually referred to as *states of nature* or *states of the world*. Thus in the insurance example the states of the world would be "a loss occurs" or "a loss does not occur".

Exercise 5.2 *A fair coin is tossed until a head appears. Denote the number of tosses to the first head by N. Assume that a pay-off of $X = 2^N$ is paid on the occurrence of the first head. Calculate the expected value of the pay-off for this game. How much would you pay to play this game? Discuss your answer.*

The problem of modelling preferences over uncertain outcomes is studied further in micro-economics.

5.2.3 Expected Utility

Utility functions are used in economics to represent preferences. For uncertain outcomes the utility function will depend on the probabilities of the states of the world. Under specified axioms of individual choice it is possible to show that individuals will rank their preferences for risks using a measure referred to as *expected utility*. Each risk can be quantified by considering the distribution of wealth in each possible state of the world. The possible states of the world need to be evaluated and probabilities assigned to these states of the world. When considering insurance, an individual's wealth will depend on whether or not they purchased an insurance policy and the premium paid for the insurance.

There are many different choices that an individual will make that will affect future wealth. These include the purchase of insurance, participation in gambling, the amount consumed and also the selection of investments for an individual's savings including superannuation.

In order to express a preference for these different alternatives it is necessary to rank them. Expected utility is one way of doing this. For all of the alternatives, the wealth in each future state of the world needs to be determined. The expected utility of wealth for each alternative is calculated and the alternative with the highest expected utility is assumed to be the preferred choice.

Assume that there are n states of the world and that for each possible state of the world we denote the wealth in that state by w_i $i = 1, 2, \dots n$. When considering insurance we could consider 2 states of the world. One would be that an event causing a loss occurs, giving rise to a claim, and the other would be that the event does not occur. In reality the states of the world will cover an infinite number of possibilities but for decision making it may be assumed that there are a finite number of outcomes to consider.

The probability that wealth will equal w_i $i = 1, 2, \ldots n$ will be the probability that a state of the world occurs with this wealth and will be denoted by

$$P\left(W = w_i\right)$$

Note that

$$\sum_{i=1}^{i=n} P\left(W = w_i\right) = 1$$

and

$$0 \leq P\left(W = w_i\right) \leq 1$$

We assume that a utility function exists that can be represented as a probability weighted sum of a function of the wealth in each possible state. Denote the utility function by $U(W)$. Expected utility is defined as

$$U(W) = E\left[v(W)\right] = \sum\nolimits_{i=1}^{i=n} P\left(W = w_i\right) v\left(w_i\right)$$

where v is assumed to be a continuous non-decreasing function.

Consider two alternatives X and Y, then under the expected utility property, an individual prefers X to Y (which will be written $X \succ Y$) if

$$U(X) > U(Y)$$

The individual is indifferent between X and Y (which will be written $X \sim Y$) if

$$U(X) = U(Y)$$

Note that the expected utility property holds for any *positive affine transformation* of v. A positive affine transformation is defined as of the form $av(w)+b, a > 0$ so that it involves multiplication by a positive number and adding a constant.

If we take such a transformation for v and calculate the expected utility we obtain

$$\begin{aligned}
& \sum\nolimits_{i=1}^{i=n} P\left(W = w_i\right)\left[av\left(w_i\right) + b\right] \\
= & \sum\nolimits_{i=1}^{i=n} \left\{aP\left(W = w_i\right) v\left(w_i\right) + P\left(W = w_i\right) b\right\} \\
= & \ a\sum\nolimits_{i=1}^{i=n} \left\{P\left(W = w_i\right) v\left(w_i\right)\right\} + b \\
= & \ aU(W) + b
\end{aligned}$$

so that we can see that the expected utility property holds for a positive affine transformation of the utility function. Any positive affine transformation also retains the same preference ordering (or ranking) as for the original utility function since

$$aU(X) + b > aU(Y) + b \quad a > 0$$

if and only if

$$U(X) > U(Y)$$

Expected utility can be validly used for ranking different risky outcomes provided a number of axioms or assumptions about individual preferences hold.

There are a number of technical assumptions required for this to hold such as that preferences are:

1. **complete:** it is assumed that all risks can be compared and ranked
2. **reflexive:** it is assumed that $X \succsim X$ so that any risk is at least as good as the risk itself
3. **transitive:** it is assumed that if $X \succsim Y$, and $Y \succsim Z$ then $X \succsim Z$.

The key assumption about preferences that allows us to use expected utility is referred to as the **independence axiom**. The independence axiom states that if X is preferred to Y, then a lottery that pays X with probability α and Z with probability $(1-\alpha)$ will be preferred to a lottery that pays Y with probability α and Z with probability $(1-\alpha)$. This means that there is independence with respect to *probability mixtures* of uncertain outcomes. Adding another risk to alternatives is assumed to not alter the ranking of these alternatives.

An important property of preferences is *risk aversion.* Preferences exhibit risk aversion when the expectation of a risk is preferred to the risk i.e. actuarially fair gambles are unacceptable.

If

$$E[W] \succ W$$

then

$$U(E[W]) = v(E[W]) > U(W) = E[v(W)]$$

If an individual prefers a gamble or risk over the certain amount equal to the expected value then they are said to be *risk lovers.* If an individual is indifferent between the expected value and a gamble or a risk then they are *risk neutral.*

Risk aversion implies that the utility function is *concave* with $\frac{\partial}{\partial W}U(W) > 0$ and $\frac{\partial^2}{\partial W^2}U(W) < 0$. To prove this we require *Jensen's inequality.* This states that for a random variable W and a function $v(W)$

$$v(E[W]) > E[v(W)] \text{ if } \frac{\partial^2}{\partial W^2}U(W) < 0$$

or

$$v(E[W]) < E[v(W)] \text{ if } \frac{\partial^2}{\partial W^2}U(W) > 0$$

Exercise 5.3 *Prove that $v(E[W]) > E[v(W)]$ if $\frac{\partial^2}{\partial W^2}U(W) < 0$ (Jensen's inequality) and comment on the meaning of the result.*

Risk preferences exhibit *risk neutrality* if $\frac{\partial^2}{\partial W^2}U(W) = 0$. If this is the case then $U(W) = aW + b$.

Exercise 5.4 *Show that if (and only if) $\frac{\partial^2}{\partial W^2}U(W) = 0$ then $U(W) = aW + b$.*

To illustrate how expected utility can be used to assess risk consider the decision to purchase an insurance policy. Assume that a loss of \$$I$ will be incurred with probability p. Assume that the amount of insurance (the sum insured) to be purchased is \$$I$ (full insurance) and that the premium for the insurance is a proportion of the amount of insurance equal to πI. Also assume that an individual has current wealth W. We can calculate the expected utility with and without the insurance as follows.

If the individual purchases full insurance then her wealth will be

$$\begin{array}{ll} W - I + I - \pi I & \text{with probability } p \text{ (a loss event occurs)} \\ W - \pi I & \text{with probability } 1-p \text{ (no loss event occurs)} \end{array}$$

The expected utility in this case will be

$$pv(W - \pi I) + (1-p)\, v(W - \pi I) = v(W - \pi I)$$

If the individual does not purchase any insurance then her wealth will be

$$\begin{array}{ll} W - I & \text{with probability } p \text{ (a loss event occurs)} \\ W & \text{with probability } 1-p \text{ (no loss event occurs)} \end{array}$$

The expected utility in this case will be

$$pv(W - I) + (1-p)\, v(W)$$

She will prefer to purchase full insurance to no insurance if

$$v(W - \pi I) > pv(W - I) + (1-p)\, v(W)$$

Example 5.5 *Assume that the probability of a loss of \$10,000 is 0.2 and that the premium for the loss is 5% of the sum insured. Assume an initial wealth of \$10,000. The utility function is assumed to be of the form*

$$v(w) = w - 0.000005w^2$$

Calculate the expected utility if full insurance is purchased and the expected utility if no insurance is purchased.

Solution 5.5 *Expected utility if full insurance is purchased will be*

$$(10,000 - 500) - 0.000005\,(10,000 - 500)^2 = 9,048.75$$

Expected utility if no insurance is purchased will be

$$0.2\,(0) + 0.8\left[10,000 - 0.000005\,(10,000)^2\right] = 7,600$$

Thus expected utility is higher if full insurance is purchased.

Exercise 5.5 *Assume that the utility function is as in the previous example i.e.* $v(w) = w - 0.000005w^2$. *Assume that you have to select one of the following two wealth distributions*

Probability	*A*	*B*
0.2	*150,000*	*100,000*
0.8	*100,000*	*80,000*

Without doing any calculations, which alternative would you prefer? Now calculate the expected utility of the two alternatives. Comment on your answer.

We can also determine the optimum amount of insurance for the individual to purchase. Assume that the individual will purchase an amount X of insurance where $0 \leq X \leq I$. Then the wealth distribution will be

$W - I + X - \pi X$ with probability p (a loss event occurs)
$W - \pi X$ with probability $1 - p$ (no loss event occurs)

and the expected utility will be

$$pv\,(W - I + X - \pi X) + (1 - p)\,v\,(W - \pi X)$$

To select the optimum amount of insurance we differentiate with respect to X and set the first derivative equal to zero to get

$$p\,(1 - \pi)\,v'\,(W - I + X - \pi X) - \pi\,(1 - p)\,v'\,(W - \pi X) = 0$$

where

$$v'\,(.) = \frac{\partial v}{\partial X}$$

We should also check that expected utility is a maximum. Thus the second derivative should be negative and we require

$$p\,(1 - \pi)^2\,v''\,(W - I + X - \pi X) + \pi^2\,(1 - p)\,v''\,(W - \pi X) < 0$$

If an individual is risk averse then $v''(.) < 0$. Since p, $(1-\pi)^2$, π^2, and $(1-p)$ are all non negative this will be negative for a risk averse individual.

If this is the case then we select X such that

$$\frac{pv'(W - I + X - \pi X)}{(1-p)\,v'(W - \pi X)} = \frac{\pi}{(1-\pi)}$$

The left hand side is the individual's trade-off between the marginal utility of an extra dollar if a loss occurs and the marginal utility of an extra dollar if no loss occurs. The ratio on the right hand side is the trade-off between the insurance market cost of an extra dollar if a loss occurs to the insurance market cost of an extra dollar if no loss occurs.

Insurance allows us to transfer wealth from the state of the world where a no-loss occurs to the state of the world where a loss occurs. It costs πX if no loss occurs to gain $(1-\pi)X$ if a loss occurs.

The results states that we optimize the amount of insurance purchased by setting our individual trade-off between wealth in the loss and no-loss states to the insurance market's trade-off between wealth in these states.

If we assume that the insurance premium is the expected value of the loss then

$$\pi = p$$

and the optimum amount of insurance is determined when

$$v'(W - I + X - \pi X) = v'(W - \pi X)$$

If an individual is risk averse then $v''(.) < 0$ which means that $v'(.)$ is decreasing and therefore it can only be equal for the same wealth value so that

$$W - I + X - \pi X = W - \pi X$$

Thus the optimal wealth is

$$X = I$$

Thus a risk averse individual will purchase full insurance if the premium is equal to the expected loss.

There are many different utility functions used in economics and finance. The quadratic utility function was used earlier. This takes the form

$$v(w) = w - \tau w^2 \quad w < W^* = \frac{1}{2\tau}$$

Another common utility function is the exponential utility function

$$v(w) = -e^{-\alpha w} \quad \alpha > 0$$

Note that

$$v'(w) = \alpha e^{-\alpha w} > 0$$

and that

$$v''(w) = -\alpha^2 e^{-\alpha w} < 0$$

Thus with $\alpha > 0$ this is the utility function of a risk-averse decision maker.

We can generalize the insurance example to allow for a continuous loss distribution. The following example illustrates how.

Example 5.6 *Assume that an individual is exposed to a loss which will occur with probability p. The loss has the following probability density (exponential)*

$$f(x) = \frac{e^{-\frac{x}{\theta}}}{\theta} \quad x \geq 0$$

Assume also that this individual has initial wealth W and an exponential utility function of the form

$$v(w) = -e^{-\alpha w} \quad \alpha > 0$$

Determine an expression for the expected utility of wealth for this individual without insurance. You should assume that $0 < \theta < \frac{1}{\alpha}$.

Solution 5.6 *The expected utility without insurance is*

$$\begin{aligned}
& (1-p)\left[-e^{-\alpha W}\right] + p\int_0^\infty -e^{-\alpha(W-x)}\frac{e^{-\frac{x}{\theta}}}{\theta}dx \\
= \; & (1-p)\left[-e^{-\alpha W}\right] + p\frac{e^{-\alpha W}}{\theta}\int_0^\infty -e^{\left(\frac{\theta\alpha-1}{\theta}\right)x}dx \\
= \; & (1-p)\left[-e^{-\alpha W}\right] + p\frac{e^{-\alpha W}}{\theta}\left[-\frac{\theta}{\theta\alpha-1}e^{\left(\frac{\theta\alpha-1}{\theta}\right)x}\right]_0^\infty \\
= \; & (1-p)\left[-e^{-\alpha W}\right] + p\frac{e^{-\alpha W}}{\theta}\frac{\theta}{\theta\alpha-1} \\
= \; & \left[(1-p) + p\frac{1}{1-\theta\alpha}\right]\left[-e^{-\alpha W}\right] \\
= \; & \left[1 + p\left(\frac{1}{1-\theta\alpha} - 1\right)\right]\left[-e^{-\alpha W}\right]
\end{aligned}$$

Exercise 5.6 *An individual is exposed to a loss which will occur with probability* 0.1. *The loss has the probability density*

$$f(x) = \frac{e^{-\frac{x}{100}}}{100} \; x \geq 0$$

The individual has initial wealth 10,000 *and an exponential utility function of the form*

$$v(w) = -e^{-0.00005w}$$

Calculate the expected utility of wealth for this individual without insurance.

5.3 Time Preference

In actuarial science the time value of money is an important factor in the valuation of cash flows from various financial security systems. A rate of interest is used to present value or discount future cash flows. The rate of interest for valuing various cash flows must be determined. In many cases the interest rate can be obtained from financial market interest rates.

One of the functions of a financial market is to provide a *price discovery* mechanism. Financial markets determine prices for buying securities that have fixed and guaranteed future pay-offs as well as securities whose future cash flows are contingent on future market conditions such as futures and options. Financial markets are either *primary* markets where securities are issued for the first time or *secondary* markets where already issued securities are traded. Secondary markets allow individuals to alter their holdings of financial instruments when they wish. Prices and interest rates of traded financial instruments are readily available from the internet, in newspapers and other financial publications.

The government bond market determines market yields to maturity for coupon paying bonds with cash flows guaranteed by the government. These cash flows can be regarded as risk-free. Other traded security markets determine yields or interest rates on securities issued by corporations. These cash flows can involve a variety of different risks. For instance loans issued by corporations involve the risk that the company will default and will not be able to pay the full amount due on the loan. This risk is referred to as *credit risk.*

Insurance and superannuation products such as life annuities and term insurance contracts do not have secondary markets. It is not possible to observe market interest rates for these products other than at the time of sale of the contracts. Rates for some companies for their life annuities and term insurance are available in various financial publications and from the internet.

To demonstrate how the time value of money is determined consider an individual who can either consume their current wealth now or at the end of a single time period. Assume that they have current wealth W and can invest for a single time period at an interest rate of r per period in a risk-free security. If they consume C_0 of their wealth now then they can invest $W - C_0$ for a single time period. The wealth at the end of the time period will be $(W - C_0)(1+r)$ and this is the amount consumed at the end of the period so that

$$C_1 = [W - C_0](1+r)$$

Note that the individual must either consume or invest their initial wealth and that they can not consume more than their initial wealth. This gives rise to a *budget constraint* as follows

$$\begin{aligned} W &= C_0 + (W - C_0) \\ &= C_0 + \frac{C_1}{(1+r)} \end{aligned}$$

Assume that utility is the total of the utility from consumption now and consumption at the end of the period. Thus utility will be

$$U(C_0) + U(C_1)$$

The individual will be assumed to select the amount of current consumption C_0 in order to maximise total utility. The maximum is determined by taking the differential of the utility function with respect to C_0 and equating this to zero. This is referred to as the *first order condition.* Doing this we obtain

$$U'(C_0) - (1+r)\,U'(C_1) = 0$$

or

$$\frac{U'(C_1)}{U'(C_0)} = \frac{1}{(1+r)}$$

Thus the optimal amount of consumption is determined so that the ratio of the marginal utility of consumption at the end of the time period to the marginal utility of consumption at the beginning of the time period is equal to $\frac{1}{(1+r)}$.

The left hand side is the marginal rate of substitution between future and current consumption. The right hand side is the present value of a future dollar at the market rate of interest. Note that $\frac{1}{(1+r)}$ is a market exchange rate between future and current consumption. Every dollar we give up in consumption at the start of the period can increase consumption at the end of the period by $(1+r)$.

Note that an amount consumption of C_1 at the end of the time period has a current value of $C_1 \frac{1}{(1+r)}$.

Example 5.7 *Assume that the utility function is*

$$U(w) = -e^{-0.005w}$$

and that an individual has initial wealth of $100,000. If the market interest rate is 8% and consumption can only take place at the start and end of the period. Determine the optimal consumption.

Solution 5.7 *We have*

$$U'(w) = 0.005e^{-0.005w}$$

and require C_0 *such that*

$$\frac{U'(C_1)}{U'(C_0)} = \frac{1}{(1.08)}$$

where $C_1 = [W - C_0](1 + r)$. *This gives*

$$\frac{0.005e^{-0.005((100{,}000 - C_0)1.08)}}{0.005e^{-0.005C_0}} = \frac{1}{(1.08)}$$

This simplifies to

$$e^{-540+0.0104C_0} = \frac{1}{(1.08)}$$

so that $C_0 = 51{,}915.68$.

An individual's time preference is determined by the marginal utility of additional consumption in the future. Individuals adjust their current consumption and hence their demand for the risk free security until their individual marginal rate of substitution between current and future consumption is equal to the present value of a dollar at the market rate. Equilibrium in the financial market is obtained when the net demand for the risk free investment is zero and all individuals have reached their optimum consumption levels. Thus the market interest rate is determined by the demand for the security and this reflects the preferences of individuals for future versus current consumption.

5.4 Marginal Utility and Pricing (more advanced)

Insurance and other financial products have uncertain future cash flows. These cash flows depend on future uncertainties or future *contingencies*. In order to determine the present value of a future uncertain amount we consider an individual with current wealth W who can purchase a financial product for price P per unit which, in one time period, has a pay-off denoted by the random variable P_1.

This financial product could be an insurance policy or a financial instrument. All that we need to be able to specify is the pay-offs in future states of the world and the probability distribution of these pay-offs. For illustration in this section we will use the simplest probability distribution for future states with two possible outcomes.

Assume that P_1 equals P_u **per unit** with probability p or P_d **per unit** with probability $(1-p)$. We also assume that there is a risk free security which pays $(1+r)$ per dollar invested. Regardless of whether the pay-off on the financial product is P_u or P_d the risk free security pays the same amount. The risk or uncertainty in this situation arises from the pay-off in the financial product and the risk free security pay-off does not depend on this uncertainty. The expected pay-off of the financial product is

$$E[P_1] = P_u p + P_d (1-p).$$

The individual consumes C_0 now and purchases X_1 units of the financial product at a price of P per unit and X_0 units of the risk free asset at a price of \$1 per unit. Her wealth in one time period will be

$$X_0(1+r) + X_1 P_u \quad \text{with probability } p$$

or

$$X_0(1+r) + X_1 P_d \quad \text{with probability } 1-p$$

where

$$X_1 = \frac{W - C_0 - X_0}{P}$$

The individual will face a budget constraint of

$$W = C_0 + X_0 + X_1 P$$

We assume that the individual selects the amount consumed now and the amount invested in the financial product and the risk free asset in order to maximize her expected utility.

The objective function for the individual is to select C_0, X_0 and X_1 to maximize

$$U(C_0) + U(X_0(1+r) + X_1 P_u)\, p + U(X_0(1+r) + X_1 P_d)(1-p)$$

subject to the budget constraint.

Denote the optimal values for these by C_0^*, X_0^* and X_1^*. At these optimal values any additional holding of the financial product or the risk free asset is sub-optimal. Assume that α additional units of the financial product are held. The objective function becomes

$$U\left(C_0^* - \alpha P\right) + \left[\begin{array}{c} U\left(X_0^*\left(1+r\right) + \left(X_1^* + \alpha\right) P_u\right) p + \\ U\left(X_0^*\left(1+r\right) + \left(X_1^* + \alpha\right) P_d\right)\left(1-p\right) \end{array}\right]$$

Since we are at the optimum, the derivative of the objective function should be zero for $\alpha = 0$. This is simply because, if we consider the objective as a function of α, then a maximum occurs when the slope of the function is zero.

Taking the derivative with respect to α gives

$$-PU'\left(C_0^* - \alpha P\right) + \left[\begin{array}{c} P_u U'\left(X_0^*\left(1+r\right) + \left(X_1^* + \alpha\right) P_u\right) p + \\ P_d U'\left(X_0^*\left(1+r\right) + \left(X_1^* + \alpha\right) P_d\right)\left(1-p\right) \end{array}\right]$$

Setting this to zero at $\alpha = 0$ gives

$$-PU'\left(C_0^*\right) + P_u U'\left(X_0^*\left(1+r\right) + X_1^* P_u\right) p + P_d U'\left(X_0^*\left(1+r\right) + X_1^* P_d\right)\left(1-p\right) = 0$$

We can solve for the price of the financial product which holds at the optimum

$$P = P_u \frac{U'\left(X_0^*\left(1+r\right) + X_1^* P_u\right)}{U'\left(C_0^*\right)} p + P_d \frac{U'\left(X_0^*\left(1+r\right) + X_1^* P_d\right)}{U'\left(C_0^*\right)}\left(1-p\right)$$

This price looks like an expected value of the pay-offs. However the original probabilities are multiplied by factors which depend on marginal utility of the optimal wealth in each state.

We can express this as an expected value. To do this we first follow the same procedure that we used for the product but for the risk free asset. If we invest α additional funds into the risk free asset at our optimum position then the value of the objective function becomes

$$U\left(C_0^* - \alpha\right) + U\left(\left(X_0^* + \alpha\right)\left(1+r\right) + X_1^* P_u\right) p + U\left(\left(X_0^* + \alpha\right)\left(1+r\right) + X_1^* P_d\right)\left(1-p\right)$$

The derivative with respect to α at $\alpha = 0$ is

$$-U'\left(C_0^*\right) + \left(1+r\right) U'\left(X_0^*\left(1+r\right) + X_1^* P_u\right) p + \left(1+r\right) U'\left(X_0^*\left(1+r\right) + X_1^* P_d\right)\left(1-p\right)$$

Setting this to zero gives

$$1 = \frac{\left(1+r\right) U'\left(X_0^*\left(1+r\right) + X_1^* P_u\right)}{U'\left(C_0^*\right)} p + \frac{\left(1+r\right) U'\left(X_0^*\left(1+r\right) + X_1^* P_d\right)}{U'\left(C_0^*\right)}\left(1-p\right)$$

Thus we see that the factors appearing in the formula for P derived above add to 1 when each of them is multiplied by $(1+r)$. These are therefore like probabilities. Now consider the expression for P derived previously. This can be written as

$$P = \frac{1}{1+r}\left[\begin{array}{c} P_u\frac{(1+r)U'\left(X_0^*(1+r)+X_1^*P_u\right)}{U'\left(C_0^*\right)}p \\ +P_d\frac{(1+r)U'\left(X_0^*(1+r)+X_1^*P_d\right)}{U'\left(C_0^*\right)}(1-p) \end{array}\right]$$

Now note that $\frac{(1+r)U'\left(X_0^*(1+r)+X_1^*P_u\right)}{U'\left(C_0^*\right)}p$ and $\frac{(1+r)U'\left(X_0^*(1+r)+X_1^*P_d\right)}{U'\left(C_0^*\right)}(1-p)$ are like probabilities, so the price can be written as

$$\begin{aligned} P &= \frac{1}{1+r}\left[P_u q + P_d(1-q)\right] \\ &= \frac{1}{1+r}E^Q\left[P_1\right] \end{aligned}$$

where $q = \frac{(1+r)U'\left(X_0^*(1+r)+X_1^*P_u\right)}{U'\left(C_0^*\right)}p$.

We can interpret this result as giving the price of the product as a "discounted expected value" of the future uncertain pay-off where adjusted probabilities are used to calculate the expected value and the expected value is present valued using the interest rate on a risk free investment. We have combined probability and time value of money, using concepts of optimization in economics, to produce a way of valuing uncertain future cash flows on financial products.

5.5 Premium Principles (more advanced)

Actuaries have developed principles that can be used to determine premiums for insurance products. The premium charged is greater than the expected value of the insurance losses. There is a loading to cover expenses and a loading to cover risk and profit. The risk loading is for the uncertainty in the insurance loss cash flows. The loadings result in a premium which is higher than the expected value of the future loss payments. The loading to cover risk is often referred to as a *risk premium*. The expected value of the future payments is called the *pure premium* or actuarially fair premium.

5.5.1 Principle of equivalence

The *principle of equivalence* states that the premium is determined so that the discounted expected value of the future premiums is equal to the discounted expected value of the loss payments and expenses. Expenses on an insurance contract occur in the form of *initial expenses* and *recurrent or renewal expenses*. The initial expenses may be a fixed amount per policy, a percentage of the sum insured or a percentage of the premium. They are incurred in selling and underwriting the policy and include the costs of establishing the policy on the insurance company's records. Renewal expenses can also be fixed amounts or a percentage of the premium. For example some government charges are levied as a percentage of the insurance premium. Fixed costs are incurred regardless of the size of the sum insured.

In order to include a loading for risk when using the principle of equivalence it is necessary to explicitly load the premium or to adjust the probabilities used in the expected values in order to include such a loading. The Esscher transform does exactly this and its use results in the *Esscher premium principle.*

5.5.2 Esscher transform

The Esscher transform is a technique for pricing insurance risk developed many years ago in actuarial science. It can be applied to both discrete and continuous loss distributions. We consider the continuous case here. The Esscher transform transforms the probabilities of a random loss $X > 0$ by multiplying the original probability density $f(x)$ by

$$\frac{e^{hX}}{E\left[e^{hX}\right]} \quad h > 0$$

This adjusted probability density is the Esscher transform of the original probability density. Note that the integral of the transformed probability density is still one i.e.

$$\begin{aligned}
\int_0^\infty \frac{e^{hX}}{E\left[e^{hX}\right]} f(x)dx &= \frac{1}{E\left[e^{hX}\right]} \int_0^\infty e^{hX} f(x)dx \\
&= \frac{E\left[e^{hX}\right]}{E\left[e^{hX}\right]} \\
&= 1
\end{aligned}$$

The premium is calculated as

$$E\left[\frac{e^{hX}}{E\left[e^{hX}\right]} X\right] = \frac{1}{E\left[e^{hX}\right]} \int_0^\infty e^{hX} X f(x)dx$$

The Esscher premium principle corresponds to the use of an exponential utility function.

Consider an individual who has obtained an optimal position with respect to her purchase of insurance with optimum wealth W^*. Assume that she has an exponential utility function given by

$$-e^{-hW} \quad h > 0$$

Consider a loss X with premium P. If she purchases an additional proportional amount α of insurance against the loss then her expected utility will be

$$U(W^*) = E\left[-e^{-h(W^* - \alpha P + \alpha X)}\right] \quad h > 0$$

At the optimum the first derivative of the utility function with respect to α will be zero for $\alpha = 0$. The first derivative is

$$\frac{\partial}{\partial \alpha} U\left(W^*\right) = E\left[\left(P - X\right) h\left(-e^{-h\left(W^* - \alpha P + \alpha X\right)}\right)\right]$$

Setting this derivative to zero for $\alpha = 0$ we get

$$E\left[\left(P - X\right) h\left(-e^{-h\left(W^*\right)}\right) = 0\right]$$

Solving gives

$$P = E\left[X \frac{\left(e^{-h\left(W^*\right)}\right)}{E\left[\left(e^{-h\left(W^*\right)}\right)\right]}\right]$$

Note that this is similar to the Esscher premium principle with a number of differences. In particular, the utility function is evaluated at the optimal wealth W^*. If future wealth was just determined by the risk under consideration then $W = -X$ since X is a loss and therefore a reduction in wealth. We would then obtain the Esscher premium.

5.6 Conclusions

This chapter has illustrated how expected utility can be used to determine preferences or rankings over alternatives. Utility is also fundamental to the valuation of risky cash flows. An excellent coverage of microeconomics can be found in Varian (1999). The material on insurance and expected utility in this chapter draws from Chapter 12 in that text. See also Chapter 4 in Panjer et al (1998) for coverage of equilibrium valuation.

5.7 Solutions to Exercises

Ex 5.1 *The risk with life annuities is that the lives will live longer than expected so that the value of payments actually made is higher than allowed for in the charge for the annuity.*

The present value of a life annuity is equal to the sum of the present value of each payment $\frac{{}_kp_x}{[1+i]^k}$. *To include a risk loading for an annuity requires a reduction in the mortality rates. A reduction in each of the* q_x *rates increases each of* $p_x = 1 - q_x$ *rates. Since the survival probability* ${}_kp_x = \frac{l_{x+k}}{l_x} = \frac{l_{x+1}}{l_x}\frac{l_{x+2}}{l_{x+1}} \cdots \frac{l_{x+k}}{l_{x+k-1}} = p_x p_{x+1} \ldots p_{x+k-1}$ *we can see that a reduction in each of the* q_x *rates increases* ${}_kp_x$. *This will increase the present value of the annuity thus including a risk loading.*

Note that in order to provide a risk loading for an annuity (as an increased charge for the product) the mortality rates will need to be increased whereas with a term life insurance which pays benefits on death the mortality rates will need to be decreased.

Ex 5.2 *Expected value of the pay-off* $\sum_1^\infty \left(\frac{1}{2}\right)^N 2^N = \sum_1^\infty 1$ *which is infinite. The chance of obtaining the large payoffs is very small. You would not pay the expected value to play this game! You would place a much lower value on the game than the expected value.*

Ex 5.3 *Consider the function* $v(x)$. *We have that*

$$v(x) \approx v(y) + (x-y)v'(y) + \frac{1}{2}(x-y)^2 v''(y)$$

Take $y = E[W]$ *and* $x = W$ *to get*

$$v(W) \approx v(E[W]) + (W - E[W])v'(E[W]) + \frac{1}{2}(W - E[W])^2 v''(E[W])$$

Now if $\frac{\partial^2}{\partial W^2}U(W) < 0$ *then* $v''(W) < 0$ *so that*

$$v(W) \leq v(E[W]) + (W - E[W])v'(E[W])$$

If we take the expected value of both sides then the LHS is $E[v(W)]$.
For the RHS we obtain

$$\begin{aligned} & E[v(E[W]) + (W - E[W])v'(E[W])] \\ = \; & E[v(E[W])] + E[(W - E[W])v'(E[W])] \\ = \; & E[v(E[W])] + v'(E[W])E[(W - E[W])] \\ = \; & E[v(E[W])] \\ = \; & v(E[W]) \end{aligned}$$

since the expected value of a constant $v(E[W])$ *is equal to the constant and the term* $E[(W - E[W])] = 0$.
To see this consider the discrete case

$$\begin{aligned} E[(W - E[W])] &= \sum p_i (W_i - E[W]) \\ &= \sum p_i W_i - \sum p_i E[W] \\ &= E[W] - E[W]\sum p_i \\ &= 0 \end{aligned}$$

A similar result applies for the case of a continuous random variable replacing the summation with integration.
Thus we have

$$v(E[W]) \geq E[v(W)]$$

This result just states that the utility of the expected value of random wealth W exceed the expected utility of the random wealth W. Thus if we are risk averse $\left(\frac{\partial^2}{\partial W^2}v(W) < 0\right)$ then we prefer the certain amount of the expected value to the uncertain risk W.

Ex 5.4 *If $\frac{\partial^2}{\partial W^2}U(W) = 0$ then integrating once gives $\frac{\partial}{\partial W}U(W) = a$ and integrating twice gives $U(W) = aW + b$ where a and b are constants. The only if follows by reversing the procedure.*

Ex 5.5 *Without doing any calculations we see that Investment A always provides a higher wealth than Investment B. Thus we should always prefer Investment A. The expected utility of the two alternatives are*
Investment A
$0.2\times(150000 - 0.000005\times 150000^2) + 0.8\times(100000 - 0.000005\times 100000^2) = 47,500$
Investment B
$0.2\times(100000 - 0.000005\times 100000^2) + 0.8\times(80000 - 0.000005\times 80000^2) = 48,400$
So Investment B appears to have higher expected utility.
This result arises from the quadratic utility function used. With the parameter 0.000005 *the quadratic utility function is increasing up to* $100,000$ *and decreasing thereafter. With this utility function an amount of \$150,000 has utility 37,500 compared with utility for an amount of \$80,000 of 48,000. To compare these alternatives, we need a properly defined utility function over the full range of wealth.*

Ex 5.6 *Expected utility of wealth without insurance is*

$$
\begin{aligned}
& 0.9\left(-e^{-0.005\times 10,000}\right) + 0.1\int_0^\infty -e^{-0.005(10000-x)}\frac{e^{-\frac{x}{100}}}{100}dx \\
= \;& 0.9\left(-e^{-0.005\times 10,000}\right) + 0.1\left(-e^{-0.005(10000)}\right)\frac{1}{100}\int_0^\infty e^{\frac{x}{200}}e^{-\frac{x}{100}}dx \\
= \;& 0.9\left(-e^{-0.005\times 10,000}\right) + 0.1\left(-e^{-0.005(10000)}\right)\frac{1}{100}\int_0^\infty e^{-\frac{x}{200}}dx \\
= \;& 0.9\left(-e^{-0.005\times 10,000}\right) + 0.1\left(-e^{-0.005(10000)}\right)\frac{1}{100}\frac{200}{1}\left[-e^{-\frac{x}{200}}\right]_0^\infty \\
= \;& [0.9 + 0.2]\left(-e^{-0.005\times 10,000}\right) \\
= \;& 1.1\left(-e^{-0.005\times 10,000}\right)
\end{aligned}
$$

Chapter 6
ACTUARIAL MANAGEMENT AND ACCOUNTING

6.1 Learning Objectives

The main objectives of this chapter are:

- to outline the main items appearing in the accounts of an insurance company and used for determining and reporting profit,
- to introduce the concept of expected profit and return on capital in insurance and
- to outline the actuarial valuation approaches to determining balance sheet values for insurance liabilities.

6.2 Profit and Surplus

Insurance companies sell products with uncertain cash flows that often extend into the future over long time periods.

These include life insurance products where the future payment is made on survival (life annuities) or on death of the life insured (term insurance, whole-of-life insurance). The companies receive premiums in return for undertaking to make future payments dependent on future survival probabilities. Claim payments could be made many years after the policy is issued.

In non-life insurance, premiums are paid to the insurance company in return for the payment of claims in respect of damage to property or personal injury from accident or natural hazards such as fire or storm. Even though these claim events can occur during a policy year, the payment of the claims may extend many years into the future. As an example for liability policies with payments for personal injury, the payment of the claim may have to wait for an assessment of the liability and the claim amount. This could involve a court procedure to determine liability. Often these court proceedings take a long time to settle. Thus payment for liability policies especially can be deferred many years into the future. As another example, consider the liability for a drug that many years in the future is discovered to cause defects. Any claims on such a liability policy will occur many years after the premium is received.

The characteristics of these products are usually that:

- a fixed premium(s) is paid to the company in advance of the claim payments,
- a large portion of the expenses of the product are incurred up-front as *acquisition costs*,
- the claim payments are uncertain and are contingent on future loss events covered by the policy, and
- the insurance company often guarantees part or all of the amount of the claim payment.

The realized profit from issuing insurance contracts can not be exactly determined until the end of the policy term after all claims and expenses have been made. For this reason actuaries determine an estimate of profit arising in a period during the life of the policy which is also referred to as *surplus*.

The revenue an insurance company receives includes premiums and investment earnings on its assets. The expenses are in respect of claims and operating expenses. The profit for a particular period for an insurance company is not just the difference between *revenue* and *expenses*. There is a mismatch in the timing of revenues and expenses for insurance contracts.

The expenses in the form of claim payments will usually be paid in future accounting periods after the premium is received. If the revenue less expenses were taken as the profit for an accounting period without adjusting for the timing difference between receipt of premium and payment of claims then a policy would show a large profit when the premium is received and a large loss if a claim is eventually paid at the time of the claim.

The profit from an insurance policy can be determined by valuing future premiums and valuing future claims and expenses. The difference between the present value of the revenue (premiums) and the expenses (claims and expenses) will be an estimate of the value of total future profit that will be earned on the policy.

In order to determine the profit or loss for an insurance company it is necessary to allocate the total profit or loss to individual accounting periods. If we value the policy liability at the start of the period and then value it at the end of the period, the difference in these values will contribute to the profit for the accounting period. Insurance company liabilities are valued by determining the value of future claims and expenses and deducting the present value of the premiums. The valuation of insurance liabilities requires the application of insurance and financial mathematics and is covered in more advanced actuarial subjects.

The premiums actually received less any claims and expenses incurred during the accounting period plus any change in the expected future profit will represent the profit from the insurance business arising during the accounting period. This is usually referred to as the *underwriting profit*. It is the profit that arises from operating the insurance business.

In order to determine total profit an allowance has to be made for the investment earnings on the assets held. In order to determine these profits it is necessary to take into profit any investment income received during the period and any realized gains or losses on any assets sold. The increase (decrease) in the asset values during the period represents unrealized gains (losses) that have accrued during the accounting period. These need to also be taken into account as profit or loss in order to determine the profit occurring during the accounting period.

Accounting Standards

Financial statements for insurance companies are required for a variety purposes. Accountants refer to *general purpose financial statements* prepared in accordance with *generally accepted accounting principles.* The aim of these statements is usually to determine the profit or loss arising in a period by allocating revenues and expenses to accounting periods. Different countries have adopted differing accounting standards for reporting the financial performance of insurance companies.

For example, *US GAAP* (Generally Accepted Accounting Principles) for insurance companies is covered in a number of Financial Accounting Standards prepared by the Financial Accounting Standards Board (FASB) in the US. In Australia, profit reporting for life insurance companies is based on the *Margin-on-Services* method. The International Accounting Standards Committee is in the process of developing an International Accounting Standard for Insurance Accounting.

The aim of accounting standards for insurance companies is to produce *realistic financial statements* including realistic profit figures and realistic values of assets and liabilities in the financial statements. Asset values are usually determined as the current market value or as a *fair value* based on the amount that the asset could be exchanged between knowledgeable and willing parties in an arm's length transaction.

Regulators of insurance companies are less concerned with realistic profit reporting and more concerned with the ability of insurance companies to meet their future claims even under adverse financial circumstances. They are interested in ensuring that the company has sufficient reserves in the form of shareholder capital, retained earnings and liability reserves on its balance sheet to provide for the solvency of the company.

Statement of Profit or Loss (Statement of Financial Performance)

If the Profit and Loss statements for an insurance company are examined then the following items will usually appear:

Revenue

Premiums - Direct premiums plus reinsurance premiums less outward reinsurance premiums

Investment earnings - Interest, dividends and realized and unrealized gains on assets

Expenses

Operating expenses
Policy acquisition expenses
Policy maintenance expenses
Investment management expenses
Interest paid on borrowings
Claims expenses
Increase in policy liabilities.

The profit will be the excess of revenue over expenses. Revenue includes the premiums received during the period by the company from the insurance policies that it issues to the public. These are referred to *direct premiums.*

An insurance company will sometimes *reinsure or underwrite* the risks of another insurance company. When an insurance company insures with another insurance company this is called *reinsurance.* The company will also sell some of its insurance risks to other insurance companies and will pay reinsurance premiums to these other companies.

Premium income is the total of direct and reinsurance premiums less any premiums the company paid to other insurance companies to reinsure its risks.

The company invests the net premiums received along with any retained earnings arising from previous profits. The company will receive investment earnings on investments. This includes interest payments on loans and bonds purchased by the company as well as dividends on shares. The capital value of assets vary as economic conditions change and this gives rise to capital gains or losses on investments. If an asset is sold for a gain or loss then this is a realized gain or loss and is included in revenue. If the value of the asset increases or decreases and the company does not sell the asset to realise the gain, then the asset will have an unrealised gain or loss. The amount of this gain or loss is the gain or loss that would occur if the company realised the asset at its market value. This is usually taken into account as profit or loss depending on whether a gain or loss would occur.

When unrealised gains or losses are brought to account as profit or loss this is usually referred to as *marking the assets to market value.*

Expenses for an insurance company include expenses incurred in selling the product and in the ongoing management of the insurance company. These expenses arise as fees, commission, salaries, telephone, rent, computing and other expenses. Claims payments are also included as expenses. Tax expenses are accounted for separately. Expenses are incurred at the time of the sale of the product. These are referred to as *policy acquisition*, *initial* or *up-front expenses.* There are also expenses incurred in maintaining the policy records of the company and managing the ongoing operations of the company. These are referred to as *policy maintenance* or *renewal expenses.*

There is also an item included in the expenses equal to the increase in the policy liabilities. Policy liabilities is the value shown in the balance sheet for the future payments expected to be made on policies less future premiums expected to be received. The change in the value of the liabilities is taken into account as an expense when determining profit or loss.

In order to allocate profit to accounting periods for a insurance policy that extends over many accounting periods it is necessary to value the liabilities of the insurance company at the end of each accounting period. The change in the value of the liabilities represents an increase or decrease in the expected profits to be paid in the future and this is change in future profits is brought to account in the profit or loss in the current accounting period. This is consistent with the accounting principle of matching revenue and expenses to the periods in which they accrue.

Exercise 6.1 *Explain what happens to the accounting profit of an insurance company if the value of the policy liabilities is overstated.*

The policy liability for an insurance company is determined as the present value of the future cash flows arising on the policies of the company. It is the present value of future expected claims (and claims expenses) less future premium receipts. The details of the valuation of policy liabilities is covered in detail in more advanced actuarial subjects.

The difference between revenue and expenses is the Operating Profit. The profit is allocated to policyholders as bonuses or reductions in premiums and to shareholders as dividends. Tax is also paid from operating profit before allocation to shareholders. Any profit remaining after allocation to policyholders, income tax and dividend payments is retained earnings which is added to the equity of the company in the Balance Sheet.

A Simple Example - Pure endowment policy

To illustrate the determination of the profit of an insurance company we will consider a simple example.

Consider a *pure endowment insurance policy* issued to a life aged (x). A pure endowment pays the *sum assured* of S on *maturity* in n periods provided the life survives. We will ignore mortality so that the payment of the sum assured will not depend on the life surviving. A premium of P per period is payable in advance until maturity. Such a product may be used to finance the repayment of a debt due in n periods or as a way of saving a sum of S over the n periods. It is a savings product where a constant amount equal to the premium is saved each period until maturity to provide for a maturity payment of S.

Example 6.1 *At age 20 how would you save \$100,000 by the time you are 40 using a life insurance product.*

Solution 6.1 *In order to do this you could purchase a pure endowment policy from a life insurance company. You would pay a regular premium over the 20 years and in 20 years time you will receive \$100,000 if you are alive. For life insurance products the premium is paid in advance i.e. at the start of each year.*

The maturity proceeds on a pure endowment policy are similar to those on a zero coupon bond. However with a zero coupon bond a single payment is made initially to purchase the bond. With the pure endowment a series of payments is made as premiums and not as a single payment.

We use the following notation for the expenses incurred on the policy. The *policy acquisition or up-front expenses* for the policy for a given sum assured are a fixed amount of I. Other expenses are per period and related to the premium. They are k per unit of the premium and a fixed amount for a given sum assured of c. For simplicity we assume that they are incurred at the start of the period. We assume that they are payable in advance at the same time as when the premium is paid on the first day of each period.

The insurance company invests the premiums received in assets. For this example we will assume that the net premiums received are invested in one year investments earning a return of r per period. This interest rate will be used to calculate the premium.

The premium to be charged is often calculated using *profit tests*. Profit tests involve projecting the future cash flows on the policy and calculating a premium that will meet the claims and expenses with reasonable certainty and provide the company with a desired profit.

By ignoring mortality, for the pure endowment policy we can use basic financial mathematics to determine the premium. The premium can be calculated using the *principle of equivalence* by equating the present value of the revenue (premiums) to the present value of the expenses (claim payment, policy acquisition and maintenance expenses).

The cashflows on the pure endowment policy including premiums and expenses are as follows:

Time	**Revenue**	**Expenses**
0	P	$I + kP + c$
1	P	$kP + c$
2	P	$kP + c$
$\vdots$	$\vdots$	$\vdots$
n	0	S

The present value of the revenue is the value of the premiums paid in advance for n periods at the assumed interest rate of r.

Recall that the present value of 1 per period payable in advance is the value of an annuity-due and is given by the formula

$$\begin{aligned}\ddot{a}_{\overline{n}|} &= \sum_{j=0}^{j=n-1}\left(\frac{1}{1+r}\right)^j \\ &= [1+r]\left[\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right] \\ &= [1+r]\,a_{\overline{n}|}\end{aligned}$$

where $a_{\overline{n}|}$ is the value of an ordinary annuity for n periods with payments in arrears.

The present value of the premium income with payments in advance will be

$$P\ddot{a}_{\overline{n}|} \text{ at rate } r$$

Example 6.2 *Determine the present value of a premium of \$100 per annum payable in advance for 10 years at an interest rate of 5% p.a.*

Solution 6.2 *Value is* $100\ddot{a}_{\overline{10}|}$ *at rate* 5%. *The value is*

$$\begin{aligned}& 100\sum_{j=0}^{j=9}\left(\frac{1}{1.05}\right)^j \\ &= 100\,[1.05]\left[\frac{1-\left(\frac{1}{1.05}\right)^{10}}{0.05}\right] \\ &= 100 \times 8.1078217 \\ &= \$810.78\end{aligned}$$

Exercise 6.2 *Calculate the present value of a premium of \$1,000 per month payable in advance for 5 years at an interest rate of 6% p.a. (nominal, monthly compounding).*

The present value of the outgoings will be the present value of the sum assured on maturity plus the present value of the expenses.

Recall that the present value of a certain payment of 1 in n periods time at an interest rate of r per period is

$$v^n = \left(\frac{1}{1+r}\right)^n$$

The present value of the sum assured payable on maturity of the policy will be

$$Sv^n \text{ at rate } r$$

Example 6.3 *Calculate the present value of \$100,000 payable in 10 years time at an effective interest rate of 5% p.a.*

Solution 6.3 *The present value is*

$$\begin{aligned} & 100,000v^{10} \text{ at 5\% p.a.} \\ = & \ 100,000\left(\frac{1}{1.05}\right)^{10} \\ = & \ 100,000 \times 0.613913 \\ = & \ \$61,391.33 \end{aligned}$$

Exercise 6.3 *Calculate the present value of \$100,000 payable in 5 years at an interest rate of 6% p.a. (nominal, monthly compounding).*

The present value of the expenses will be the present value of the acquisition expenses of I, plus the present value of the maintenance expenses that are k per unit of the premium P, plus the fixed amount of c. The acquisition expenses are a one-off payment at the current time so they have present value equal to I. The per unit of premium expenses are equal to kP and have present value $kP\ddot{a}_{\overline{n}|}$. The fixed amount per period expenses of c have present value $c\ddot{a}_{\overline{n}|}$. Thus the present value of total acquisition and maintenance expenses will be

$$I + kP\ddot{a}_{\overline{n}|} + c\ddot{a}_{\overline{n}|}$$

Using the principle of equivalence the premium is calculated by equating the present value of the revenue to the present value of the expenses. Thus we have

$$P\ddot{a}_{\overline{n}|} = Sv^n + I + kP\ddot{a}_{\overline{n}|} + c\ddot{a}_{\overline{n}|}$$

Solving for P gives

$$P = \frac{1}{(1-k)}\left[\frac{Sv^n}{\ddot{a}_{\overline{n}|}} + \frac{I}{\ddot{a}_{\overline{n}|}} + c\right]$$

Example 6.4 *A pure endowment policy with a 5 year maturity is issued with a sum assured of \$100,000. Initial acquisition expenses are \$5,000. Other expenses are a fixed amount of \$20 on payment of each premium and 1% of the premium. The interest rate is assumed to be 5% p.a. effective. The premium is a level annual amount paid in advance. Ignore mortality. Calculate the premium allowing for these expenses.*

Solution 6.4 *The present value of the sum assured is*

$$\begin{aligned} & 100,000v^{5} \text{ at 5\%} \\ = & \ 100,000\left(\frac{1}{1.05}\right)^{5} \\ = & \ 78,352.62 \end{aligned}$$

We also have that $\ddot{a}_{\overline{5}|}$ *at 5%* $=[1.05]\left[\frac{1-\left(\frac{1}{1.05}\right)^5}{0.05}\right] = 4.54595$. *The expenses are* $I = 5{,}000$, $c = 20$, *and* $k = 0.01$.
The premium is

$$\begin{aligned} & \frac{1}{(0.99)}\left[\frac{78{,}352.62}{4.54595} + \frac{5{,}000}{4.54595} + 20\right] \\ = & \frac{1}{(0.99)}[17{,}235.69 + 1{,}099.88 + 20] \\ = & 18{,}540.98 \end{aligned}$$

Exercise 6.4 *A pure endowment policy with a 5 year maturity is issued with a sum assured of $100,000. Initial acquisition expenses are $5,000. Other expenses are a fixed amount of $50 on payment of each premium and 1.5% of the premium. The interest rate is assumed to be 6% p.a. effective. The premium is a level annual amount paid in advance. Ignore mortality. Calculate the premium allowing for these expenses.*

To simplify the details for the remainder of this chapter we will ignore maintenance expenses and only include acquisition expenses. In this case the premium on a pure endowment allowing for only acquisition expenses would be determined using

$$P\ddot{a}_{\overline{n}|} = Sv^n + I$$

so that

$$P = \frac{Sv^n}{\ddot{a}_{\overline{n}|}} + \frac{I}{\ddot{a}_{\overline{n}|}}$$

Example 6.5 *A pure endowment policy with a 5 year maturity is issued with a sum assured of $100,000. Initial acquisition expenses are $5,000. There are no other expenses and the interest rate is assumed to be 5% p.a. effective. The premium is a level annual amount paid in advance. Ignore mortality. Calculate the premium allowing for the acquisition expenses.*

Solution 6.5 *The present value of the sum assured is as before*

$$\begin{aligned} & 100{,}000v^5 \text{ at } 5\% \\ = & 78{,}352.62 \end{aligned}$$

The acquisition expenses are $I = 5{,}000$ *so that the premium is*

$$\begin{aligned} & \left[\frac{78{,}352.62}{4.54595} + \frac{5{,}000}{4.54595}\right] \\ = & 18{,}335.58 \end{aligned}$$

Note that in this case the premium can be considered as having two components. The first component is to provide the sum assured at maturity and equals

$$\frac{Sv^n}{\ddot{a}_{\overline{n|}}}$$

This amount when accumulated at interest will provide the sum assured of S at maturity. To demonstrate this is the case we need to introduce some more actuarial notation for accumulations.

We have already defined the actuarial symbols representing the present value of an annuity and an annuity due. For accumulated values we need to consider the future values rather than the present values. Consider an annuity of 1 per period payable in arrears for n periods. The future or accumulated value of these n payments at an interest rate of r per period on the date of the last payment is

$$\begin{aligned} & (1+r)^{n-1} + (1+r)^{n-2} + \ldots + 1 \\ = & \sum_{j=0}^{j=n-1} (1+r)^j \\ = & \frac{(1+r)^n - 1}{r} \end{aligned}$$

The actuarial notation for this future value is

$$s_{\overline{n|}}$$

Now consider the future or accumulated value of an annuity of 1 per period payable in **advance** over n periods. The future or accumulated value of these payments at an interest rate of r per period at the end of n periods will be

$$\begin{aligned} & (1+r)^n + (1+r)^{n-1} + \ldots + (1+r) \\ = & (1+r) \sum_{j=0}^{j=n-1} (1+r)^j \\ = & (1+r)\, s_{\overline{n|}} \end{aligned}$$

The notation for this value is $\ddot{s}_{\overline{n|}}$.

Thus the accumulated value on the maturity date of an annuity payable in advance of $\frac{Sv^n}{\ddot{a}_{\overline{n}|}}$ per period will be

$$\begin{aligned}
& \frac{Sv^n}{\ddot{a}_{\overline{n}|}} \ddot{s}_{\overline{n}|}. \\
= & \frac{Sv^n}{\frac{(1+r)(1-v^n)}{r}} (1+r) \frac{(1+r)^n - 1}{r} \\
= & Sv^n \frac{(1+r)^n - 1}{1 - v^n} \\
= & S \frac{1 - v^n}{1 - v^n} \\
= & S
\end{aligned}$$

The other component in the premium is a charge to cover the acquisition expenses of $\frac{I}{\ddot{a}_{\overline{n}|}}$. This is equivalent to a repayment for a loan of I with repayments made in advance. The present value of the loan repayments is

$$\begin{aligned}
& \frac{I}{\ddot{a}_{\overline{n}|}} \ddot{a}_{\overline{n}|} \\
= & I
\end{aligned}$$

The policy could be considered as a liability for the sum assured plus a loan to the policyholder for the acquisition expenses. The portion of the premium payments of $\frac{I}{\ddot{a}_{\overline{n}|}}$ are sometimes treated as an asset when accounting for insurance company liabilities. If the acquisition expenses are treated as a loan to the policyholder to be repaid from the premium then the value of the outstanding repayments can be treated as an asset of the company.

To consider the determination of profit we will continue with the previous example.

Example 6.6 *A pure endowment policy with a 5 year maturity is issued with a sum assured of \$100,000. Initial acquisition expenses are \$5,000. There are no other expenses and the interest rate is assumed to be 5% p.a. effective. Mortality is ignored. The premium is* \$18,335.58 *payable in advance. Determine the profit or loss on this policy for each year.*

Solution 6.6 *In the first year the premium revenue will be* $18,335.58$ *and the expenses will be the acquisition expenses of* $5,000$. *The net cash flow at the start of the policy will be* $18,335.58 - 5,000 = 13,335.58$ *which is invested in interest earning assets.*
The investment earnings for the first year will be $0.05 \times 13,335.58 = \$666.78$.
The value of the assets at the start of the period before the policy commences is zero.

The value of the assets at the end of the period will be the initial net cash flow plus interest or $13,335.58 + 666.78 = 14,002.35$*. There are no capital gains or losses on the assets to take into account.*
The value of the policy liability at the start of the first period is zero. This can be confirmed by present valuing the future sum assured adding the initial expenses and subtracting the present value of the premiums. The value is

$$\begin{aligned} & 100,000v^5 + 5,000 - 18,335.58\ddot{a}_{\overline{5}|} \text{ at 5\% p.a.} \\ = \ & 78,352.62 + 5,000 - 18,335.58 \times 4.54595 \\ = \ & 0 \end{aligned}$$

The value at the end of the period will be the present value of the sum assured and any future expenses less the present value of the premiums remaining to be paid. Since there are no maintenance expenses assumed, this is

$$\begin{aligned} & 100,000v^4 - 18,335.58\ddot{a}_{\overline{4}|} \text{ at 5\% p.a.} \\ = \ & 82,270.25 - 18,335.58 \times 3.72325 \\ = \ & 14,002.35 \end{aligned}$$

Now the profit for the first year will be

Premium revenue + interest earnings - acquisition expenses - increase in policy liability

which is

$$\begin{aligned} & 18,335.58 + 666.78 - 5,000 - (14,002.35 - 0) \\ = \ & 0 \end{aligned}$$

Since the company did not include any profit loading in the premium we would expect the profit to be zero, as it is. Note that if we did not allow for the increase in the liability value then we would report an incorrect profit of

$$\begin{aligned} & \textit{Premium revenue + interest earnings - acquisition expenses} \\ = \ & 14,002.35 \end{aligned}$$

Now proceed to the second year.
At the start of the year a premium of $18,335.58$ *is received.*
There are no expenses.
The investment earnings will be 5% on the total of the value of the assets at the end

of the first period plus the premium paid at the start of the period. Thus interest will be $0.05 \times (18,335.58 + 14,002.35) = 1,616.90$.
There are no gains or losses on the assets either realised or unrealised. The assets are assumed to only pay interest each year like a savings account so there are no capital gains or losses.
The value of the assets at the end of the period will be the premium received at the start of the period plus the assets at the start of the year plus interest for the year or $18,335.58 + 14,002.35 + 1,616.90 = 33,954.83$.
The value of the liabilities at the start of the period is $14,002.35$ *and the value at the end of the period will be*

$$\begin{aligned} & 100,000v^3 - 18,335.58\ddot{a}_{\overline{3}|} \text{ at 5\% p.a.} \\ = \; & 86,383.76 - 18,335.58 \times 2.85941 \\ = \; & 33,954.83 \end{aligned}$$

The profit for the second year will be

Premium revenue + interest earnings - increase in policy liability

which is

$$\begin{aligned} & 18,335.58 + 1,616.90 - (33,954.83 - 14,002.35) \\ = \; & 0 \end{aligned}$$

We see again that the profit arising during the year is zero.
We would expect this to occur in each year since the premium has been determined so that its value is exactly equal to the value of the sum assured and the initial expenses. Thus total profit expected on this product is zero. This means that the profit arising in any period will also be zero.
If we proceed for each of the remaining years the following figures are obtained

Income	**Premium**	**Interest**	**Expenses**	**Claims**	**Liability increase**	**Profit**
Year 1	$18,335.58	$666.78	$5,000.00	$0.00	$14,002.35	-$0.00
Year 2	$18,335.58	$1,616.90	$0.00	$0.00	$19,952.47	$0.00
Year 3	$18,335.58	$2,614.52	$0.00	$0.00	$20,950.10	-$0.00
Year 4	$18,335.58	$3,662.02	$0.00	$0.00	$21,997.60	$0.00
Year 5	$18,335.58	$4,761.90	$0.00	$100,000.00	-$76,902.52	$0.00

Note that in the final year the value of the liability is

$$\begin{aligned} & 100,000v^1 - 18,335.58\ddot{a}_{\overline{1}|} \text{ at 5\% p.a.} \\ = \; & 95,238.10 - 18,335.58 \times 1.0000 \\ = \; & 76,902.52 \end{aligned}$$

and the liability is zero at the end of the year. Thus the liability decreases by $76,902.52$ *during the final year. The profit in the final year is*

$$\begin{aligned} & \textit{Premium revenue + interest earnings – claims - increase in policy liability} \\ = & \ 18,335.58 + 4,761.90 - 100,000 - (0 - 76,902.52) \\ = & \ 0 \end{aligned}$$

The example illustrated how profit can be determined on an insurance product and the importance of the liability valuation in allocating profits to each year. In practice the interest rates and expenses in the future will vary and so the profit will not actually be zero in future years.

Exercise 6.5 *A pure endowment policy with a 5 year maturity is issued with a sum assured of \$100,000. Initial acquisition expenses are \$5,000. There are no other expenses and the interest rate used to calculate the premium is 5% p.a. Assume that there is no mortality. The premium is* $\$18,335.58$ *payable in advance. Assume that the interest rates earned in the first 3 years are 5% p.a. and then increases to 6% p.a. for the final two years.*
Determine the profit on this policy for each year assuming that the liability value calculated at the end of each year is determined using 5% p.a.
Also determine the profit for each year assuming that the policy liability value at the end of the year is determined using the same interest as is earned on the assets during the year.
Comment on your answers.

Policies with Uncertain Claims

So far we have considered a policy where the future claim payment (sum assured) is known. Insurance policies usually cover future claims where the claim occurrence and or amount of the claim is a random variable. In this case the future profit will also be a random variable. To illustrate how this affects profit, consider a one year insurance policy covering a fixed loss of L which can occur with probability q. The liability for this loss at the start and end of the year will be zero so it is not necessary to consider the change in the liability value in determining profit.

Assume that if a claim occurs then it will be paid at the end of the year and that no interest is earned. The premium is payable in advance for an amount of P.

The profit on this policy will be a random variable. It will equal the premium income P minus the claims payment L with probability q or zero with probability $(1-q)$.
The profit at the end of the year will either be

$$P - L$$

if a claim occurs, or

$$P$$

if a claim does not occur.

The expected profit at the start of the year will be

$$\begin{aligned} & q\left[P-L\right]+(1-q)\,P \\ = \; & P-qL \end{aligned}$$

At the end of the accounting period the profit results will report the actual claims less premium receipts. This amount will depend on the claims that actually occur. Thus insurance companies with liabilities with highly variable losses will report highly variable profits.

Loading Premiums for Profit

In practice, insurance companies will charge a premium equal to the expected value of the claims plus a loading for profit. They also raise capital from investors to provide funds to cover the risks involved in running an insurance company. The more variable the claims then the greater will be the requirement for capital in order for the insurance company to meet claim payments in adverse circumstances. The loading for profit has to provide a return on capital to the investors.

Insurance companies can reduce the variability of claims, and hence the need for capital, by underwriting a large number of independent risks. This will reduce the variability of expected claims and hence reduce the variability of expected profits. This results from the law of large numbers which states that the average claim will be less variable for a large number of independent risks. Insurance companies can also reinsure part of their claims by purchasing reinsurance from other insurance companies to cover the case if the total claims exceed a specified amount. This amount is determined by considering the total capital and premium income available to meet the company's claims.

One method of loading a premium is to use the Esscher premium principle to calculate the premium. The Esscher premium is calculated using

$$P=E\left[\frac{e^{hL}}{E\left[e^{hL}\right]}L\right]$$

where h is a parameter and L is the random loss.

In the simple example where a claim of L can occur with probability q we have

$$\begin{aligned} E\left[e^{hL}\right] &= qe^{hL}+(1-q)\,e^{h0} \\ &= qe^{hL}+(1-q) \end{aligned}$$

This gives the Esscher premium of

$$\begin{aligned} P &= \frac{qe^{hL}}{qe^{hL} + (1-q)} L + \frac{(1-q)}{qe^{hL} + (1-q)} 0 \\ &= \frac{qe^{hL}}{qe^{hL} + (1-q)} L \end{aligned}$$

Example 6.7 *Assume that a company has sold a policy under which the probability of a claim is $q = 0.05$ and the amount of a claim will be $L =$\$100,000 if a claim occurs. Assume that the company calculates the premium using the Esscher premium principle with $h = 0.00001$. Calculate the expected loss and the Esscher premium.*

Solution 6.7 *The expected loss is*

$$0.05 \times 100,000 + 0.95 \times 0 = 5,000$$

The Esscher premium is

$$\begin{aligned} & \frac{0.05e^{0.00001 \times 100,000}}{0.05e^{0.00001 \times 100,000} + 0.95} \times 100,000 + \frac{0.95}{0.05e^{0.00001 \times 100,000} + 0.95} \times 0 \\ = & \frac{0.135914}{1.085914} \times 100,000 \\ = & 0.125161 \times 100,000 \\ = & 12,516.10 \end{aligned}$$

Note how the Esscher premium is similar to an expected value but with an increased probability that a loss occurs.

Note that the premium exceeds the expected loss. The addition to the expected value is referred to as a *loading*. The loading is to provide a profit for taking the risks in running the insurance business.

Exercise 6.6 *Assume that a company has sold a policy under which the probability of a claim is $q = 0.01$ and the amount of a claim will be $L =$\$100,000 if a claim occurs. Assume that the company calculates the premium using the Esscher premium principle with $h = 0.00001$. Calculate the Esscher premium.*

Capital, Risk and Profit

Insurance companies usually aim to remain solvent and pay the claims as they occur. In order to do this they raise capital from investors. In the simple example, the company will always be able to pay the claim amount of L if it has capital from investors of an amount of

$$L - P$$

If investors subscribe this amount of money to the insurance company as capital (shares) then the capital along with the premium from the policyholder will be sufficient to pay the claim. The investors will lose all of their capital if a claim does occurs. If no claim occurs then they will retain their capital plus earn the premium as profit.

The profit will be

$$P - L \text{ with probability } q$$

or

$$P \text{ with probability } (1 - q)$$

The expected return to the investors will be the expected profit divided by the amount invested which is

$$\begin{aligned} & \frac{q[P-L] + (1-q)P}{L-P} \\ = & \frac{P - qL}{L - P} \end{aligned}$$

Note that the expected profit is the premium minus the expected loss from the claim. If the premium charged exceeds the expected loss then the company will expect to make a profit.

Risk averse investors will be prepared to pay more than the expected loss to avoid a risk so that they will usually be prepared to pay a profit loading to the insurance company.

Example 6.8 *Assume that a company has sold a policy under which the probability of a claim is $q = 0.05$ and the amount of a claim will be L =\$100,000 if a claim occurs. The premium charged is \$$12,516.10$. Assume that this is the only policy that the insurance company has sold. Determine the amount of capital required to ensure that the company will always pay the claim. Determine the expected return to shareholders.*

Solution 6.8 *The company will require*

$$100,000 - 12,516.10 = 87,483.90$$

in capital to cover the claim.
The expected profit for the shareholders will be

$$0.05 \times -87,483.90 + 0.95 \times 12,516.10 = 7,516.10$$

The expected returns to the shareholders will be

$$\frac{7,516.10}{87,483.90} = 0.0859$$

Thus the expected return will be 8.59%.

Exercise 6.7 *Assume that a company has sold a policy under which the probability of a claim is* $q = 0.01$ *and the amount of a claim will be* $L =$*\$100,000 if a claim occurs. Assume that the company charges a premium equal to the expected loss plus a 30% loading. Assume that this is the only policy that the insurance company has sold. Determine the amount of capital required to ensure that the company will always pay the claim. Determine the expected return to shareholders.*

6.3 Balance Sheet

There are two main types of insurance company. They are the *mutual company* and the *shareholder company.*

A *mutual company* is owned by the policyholders. All the profits and losses from running the business are owned by the policyholders in a mutual company.

A *shareholder company* is owned by investors who subscribe capital to the company in return for a profit. In a shareholder company some of the profits are allocated to policyholders if the policyholder is entitled to a share of the profits in the form of bonuses.

Financial statements for both types of company are very similar except that the shareholder company will have share capital as part of its equity.

We have seen that in order to determine the profit of an insurance company it is necessary to value the insurance policy liabilities since the change in the value of the liabilities contributes to the profit arising during an accounting period. The value is determined as the present value of the future claims and expenses less the present value of the future premiums. The value of the liability and the value of the assets of the insurance company appear in the balance sheet of the company.

The Balance sheet or Statement of financial position for a shareholder insurance company will contain the following items:

Assets
Cash

Loans and Bonds (Debt securities)
Equity security investments
Property investments
Operating assets
Debtors (outstanding premiums)

Liabilities

Provisions (for tax and employee benefits)
Borrowings
Policy Liabilities (Future policy benefits plus future expenses plus future profit margins less future charges for acquisition expenses less future premiums).

Equity

Share capital
Retained profits for shareholders.

The investment assets (loans, bonds, equities and property) are usually valued at market value for insurance accounting purposes. Operating assets are valued at historical cost less depreciation.

The policy liabilities are valued at present values using assumptions about future claims, expenses, and interest rates. There are a variety of techniques used by actuaries to value the policy liabilities.

6.3.1 Actuarial valuation

Actuarial liability values are usually based on the prospective or future expected cash flows on the insurance company's policies. They are sometimes referred to as *prospective reserves*.

They can also be valued by accumulating the premiums less claims and expenses from the date the policy was issued to the valuation date. Such a reserve is called a *retrospective reserve*.

The retrospective reserve is similar to historical cost accounting where assets are recorded in the accounts at their original cost. No account is taken of the changes in market value using historical cost accounting. The changes in the market value of an asset reflect changes in future expected conditions. A retrospective valuation does not take into account future conditions if they have changed significantly from the date the insurance policy was sold.

When valuing the future claims, expenses and premiums it is necessary to make assumptions about the future interest rates, expenses and claim probabilities. If these assumptions are based on the best estimate of what these are likely to be in the future then the liability is referred to as a *best estimate liability*.

If the assumptions used in the actuarial valuation of the liabilities are designed to overstate the liability value then these are referred to as *conservative assumptions*. Overstating the liability value will decrease the profit reported during an accounting period since the change in the liability value is a reduction of profit. This means that profit is withheld to future accounting periods.

6.3.2 Solvency

Actuaries have traditionally been concerned with ensuring that insurance companies, and other financial security systems are able to meet their obligations with reasonable certainty. The extent to which a company can meet its future obligations is referred to as the solvency of the company.

Regulators are also concerned with ensuring that insurance companies and superannuation funds will be able to meet their future obligations. In order to do this it is necessary to assess the probability that the company will not be able to meet its claims in the future. This probability is called the *probability of ruin.* If the company uses conservative valuation assumptions that overstate the policy liabilities in its balance sheet then additional reserves are retained on the balance sheet as part of the policy liabilities. This will increase the funds available to meet future obligations.

Solvency can also be enhanced by additional subscription of capital by shareholders in a shareholder company or by retaining profits and not distributing as much as bonuses or profits to policyholders in a mutual company.

6.4 Conclusions

This chapter has introduced the concept of profit in insurance and shown how it is necessary to value the future claim payments, expenses and premiums in order to determine the profit arising during an accounting period.

The main items shown in the profit and loss account and the balance sheet for an insurance company were set out and briefly described.

Methods for determining premiums to allow for expected profits were introduced.

6.5 Solutions to Exercises

Ex 6.1 *If the value of the policy liabilities is overstated then this means the expense item equal to the increase in the policy liabilities is higher and the accounting profit for the current year is reduced. Thus current year profit is understated.*

Ex 6.2 *Note that we must use a monthly time period. The interest rate must be a monthly effective rate. We are given a nominal rate with monthly compounding so we divide this rate by 12 to get the monthly effective rate. Present value of the premium will be*

$$1000\ddot{a}_{\overline{60|}} \text{ at rate } \frac{6}{12}\% \text{ or } 0.5\%$$

$$\begin{aligned}
&= 1000\sum_{j=0}^{j=59}\left(\frac{1}{1.005}\right)^j \\
&= 1000\,[1.005]\left[\frac{1-\left(\frac{1}{1.005}\right)^{60}}{0.005}\right] \\
&= 1000 \times 51.98419 \\
&= 51,98.4.19
\end{aligned}$$

Ex 6.3 *Present value is at a rate which is nominal with monthly compounding. We must work in months with a monthly effective rate. Note that the time period and the effective interest rate per period must always be consistent. Present value is*

$$\begin{aligned}
& 100,000v^{60} \;\; \textit{at 0.5\% p.a.} \\
&= 100,000\left(\frac{1}{1.005}\right)^{60} \\
&= 100,000 \times 0.741372196 \\
&= \$74,137.22
\end{aligned}$$

Ex 6.4 *In this case we have an annual effective interest rate. The present value of the sum assured is*

$$\begin{aligned}
& 100,000v^{5} \;\; \textit{at 6\%} \\
&= 100,000\left(\frac{1}{1.06}\right)^{5} \\
&= 74,725.82
\end{aligned}$$

Present value of expenses will be

$$5000 + 0.015P\ddot{a}_{\overline{5|}} + 50\ddot{a}_{\overline{5|}} \;\; \textit{using 6\% interest rate}$$

We have $\ddot{a}_{\overline{5|}}$ *at 6%* $=[1.06]\left[\frac{1-\left(\frac{1}{1.06}\right)^5}{0.06}\right] = 4.465106.$
The principle of equivalence determines the premium using

$$P\ddot{a}_{\overline{5|}} = 100,000v^5 + 5000 + 0.015P\ddot{a}_{\overline{5|}} + 50\ddot{a}_{\overline{5|}}$$

$$\begin{aligned}
P &= \frac{1}{0.985}\left[\frac{74,725.82}{4.465106} + \frac{5000}{4.465106} + 50\right] \\
&= 18,177.97
\end{aligned}$$

Ex 6.5 *We note that this is the same example given in the text except that in Year 4 and 5 the interest rate earned is 6% p.a. For the first 3 years the profit will be the same as previously.*

Time	*Liability Value 5%*
0	$\begin{bmatrix} 100,000v^5 + 5,000 - 18,335.58\ddot{a}_{\overline{5}\rceil} \\ = 78,352.62 + 5,000 - 18,335.58 \times 4.54595 \\ = 0 \end{bmatrix}$ *at 5% p.a.*
1	$\begin{bmatrix} 100,000v^4 - 18,335.58\ddot{a}_{\overline{4}\rceil} \\ = 82,270.25 - 18,335.58 \times 3.72325 \\ = 14,004.35 \end{bmatrix}$ *at 5% p.a.*
2	$\begin{bmatrix} 100,000v^3 - 18,335.58\ddot{a}_{\overline{3}\rceil} \\ = 86,383.76 - 8,335.58 \times 2.85941 \\ = 33,954.83 \end{bmatrix}$ *at 5% p.a.*
3	$\begin{bmatrix} 100,000v^2 - 18,335.58\ddot{a}_{\overline{2}\rceil} \\ = 90,702.95 - 8,335.58 \times 1.95238 \\ = 54,904.92 \end{bmatrix}$ *at 5% p.a.*
4	$\begin{bmatrix} 100,000v^1 - 18,335.58\ddot{a}_{\overline{1}\rceil} \\ = 95,238.10 - 8,335.58 \times 1.00000 \\ = 76,902.52 \end{bmatrix}$ *at 5% p.a.*
5	0

Time	*Liability Value changed interest rates*
0	$\begin{bmatrix} 100,000v^5 + 5,000 - 18,335.58\ddot{a}_{\overline{5}\rceil} \\ = 0 \end{bmatrix}$ *at 5% p.a.*
1	$\begin{bmatrix} 100,000v^4 - 18,335.58\ddot{a}_{\overline{4}\rceil} \\ = 14,004.35 \end{bmatrix}$ *at 5% p.a.*
2	$\begin{bmatrix} 100,000v^3 - 18,335.58\ddot{a}_{\overline{3}\rceil} \\ = 33,954.83 \end{bmatrix}$ *at 5% p.a.*
3	$\begin{bmatrix} 100,000v^2 - 18,335.58\ddot{a}_{\overline{2}\rceil} \\ = 54,904.92 \end{bmatrix}$ *at 5% p.a.*
4	$\begin{bmatrix} 100,000v^1 - 18,335.58\ddot{a}_{\overline{1}\rceil} \\ = 94,339.62 - 18,335.58 \times 1.00000 \\ = 76,004.05 \end{bmatrix}$ *at 6% p.a.*
5	0

The interest earnings will change in the last two years. At the end of year 3 the value of the assets is \$54,904.92 (this is the sum of the premiums and interest less initial expenses for the first three years - check it). Interest for the 4th year will be (18,335.58+54,904.92)$\times 0.06$ = *4394.43. Similarly for the last year the value of the assets at the end of the year will be 18,335.58+54,904.92+4394.43 = 77,634.93. The interest for the last year will be (18,335.58+77,634.93)*$\times 0.06$ = *5758.23.*

The profit for each year if the liabilities are calculated at 6% for the last two years will be

Income	Premium	Interest	Expenses	Claims	Liability increase	Profit	Assets (end year)
Year 1	$18,335.58	$666.78	5000	0	$14,002.35	-$0.00	$14,002.35
Year 2	$18,335.58	$1,616.90	0	0	$19,952.47	$0.00	$33,954.83
Year 3	$18,335.58	$2,614.52	0	0	$20,950.10	-$0.00	$54,904.92
Year 4	$18,335.58	$4,394.43	0	0	$21,997.60	$732.40	$77,634.93
Year 5	$18,335.58	$5,758.23	0	100000	-$76,902.52	$996.33	$101,728.73

Note that the profit is the excess interest earned during the year i.e. in year 4 the profit is (0.06-0.05)$\times(18,335.58 + 54,904.92) = 732.40$ *and in the last year it is (0.06)*$\times(18,335.58 + 77,634.93) - (0.05) \times (18,335.58 + 76,902.52) = 5758.23 - 4761.90 = 996.33$.
If the interest rate used to calculate the liability at the end of the last two years is increased to 6% then the following results are obtained

Income	Premium	Interest	Expenses	Claims	Liability increase	Profit	Assets (end year)
Year 1	$18,335.58	$666.78	5000	0	$14,002.35	-$0.00	$14,002.35
Year 2	$18,335.58	$1,616.90	0	0	$19,952.47	$0.00	$33,954.83
Year 3	$18,335.58	$2,614.52	0	0	$20,950.10	-$0.00	$54,904.92
Year 4	$18,335.58	$4,394.43	0	0	$21,099.13	$1,630.88	$77,634.93
Year 5	$18,335.58	$5,758.23	0	100000	-$76,004.05	$97.85	$101,728.73

We see that most of the profit now occurs in Year 4. The profit figure in Year 5 is the interest at 6% on the profit from the previous year i.e. 0.06$\times 1,630.88 = 97.85$. *Note that the total profit is the same i.e. 732.40+996.33 = 1,728.73 and 1630.88+97.85 = 1,728.73. Thus when we change our interest rate to value the future liabilities when interest rates change we bring into profit in the CURRENT year profit in FUTURE years (in this case excess interest) even though we are not sure that we will earn the higher interest rate in the future.*

Ex 6.6 *The Esscher premium is*

$$\begin{aligned}
& \frac{0.01e^{0.00001\times 100,000}}{0.01e^{0.00001\times 100,000}+0.99} \times 100,000 + \frac{0.99}{0.01e^{0.00001\times 100,000}+0.99} \times 0 \\
= \;& \frac{0.0271828}{1.0171828} \times 100,000 \\
= \;& 0.026724 \times 100,000 \\
= \;& 2,672.36
\end{aligned}$$

which compares with the expected loss of 1,000.

Ex 6.7 *The expected claim cost is*

$$0.01 \times 100,000 = 1,000$$

The premium charged will be

$$1.3 \times 1,000 = 1,300$$

which is a 30% loading on the expected cost.
The amount of capital to ensure that the company will always pay the claim will be

$$100,000 - 1,300 = 98,700$$

The expected return to shareholders is

$$\frac{0.01 \times (-98,700) + 0.99 \times 1,300}{98,700} = \frac{300}{98,700} 0.00304$$

or 0.3% (not so good - so shareholders would require a higher loading to invest in this company with this single risk).

Chapter 7
RISK MANAGEMENT AND FINANCIAL SECURITY SYSTEMS

7.1 Learning Objectives

The main objectives of this chapter are:

- to introduce the concepts of risk pooling and diversification
- to outline the use of risk classification, rating and experience adjustments in risk management, and
- to briefly introduce the concepts of actuarial soundness and ruin.

7.2 Actuarial Management of Financial Security Systems

Actuaries have a long history of involvement in the *risk management* and *financial management* of financial security systems. These systems include life insurance companies, general insurance companies, superannuation and pension funds, health insurance companies as well as social security systems.

Life insurance companies sell products that provide for security in the event of death or disability as well as products to provide savings for retirement or other long term future needs.

General insurance companies, also called *property and casualty companies*, provide for financial security in the event of property loss or liability for injury.

Superannuation funds provide for financial security in retirement by accumulating savings during an individual's working life from both individuals and employers.

Health insurance provides cover for the cost of medical and hospital treatment.

Social security benefits are provided by government to cover unemployment, aged and other pension, disability benefits as well as publicly provided hospital and medical care.

These financial security systems require actuarial assessment and risk management in order to ensure they fulfill their future obligations and to ensure fair treatment between the individuals covered by the systems. These systems pool risks for groups of individuals. This pooling of risk results in an overall increase in the welfare of the individual members since large uncertain losses are averaged across groups of individuals.

Risk management requires the assessment, underwriting and classification of risks in order to charge a fair premium for the risk involved. Where the risk can not be precisely assessed, *experience rating* and profit sharing schemes are often used to ensure that the cost of the risk reflects an individual's claim experience.

The assessment of the future liabilities of these financial security systems and the establishment of investment policy and objectives is an important role of the actuary in ensuring the financial soundness of these schemes.

7.3 Pooling of Risk and Diversification

Pooling of risk is a major function of social security systems. By participating in a financial security system, individuals can reduce the risk they face by replacing large individual losses that they would not otherwise be able to bear with smaller more certain losses shared across all the participants in the systems equal to the average loss.

Individuals pool common but independent risks in a financial security system under which they will be compensated for their individual losses. In return they are charged the average cost of the total risks for the pool plus the costs of running the system including a return on any capital from external funds such as shareholders funds in a public company.

We have already seen that for risk averse individuals their expected utility is higher when a large uncertain loss is replaced by a certain amount equal to the expected value of the loss. Thus insurance pooling systems increase the expected utility of the individuals in the pool and benefit the community as a whole.

In a life insurance fund, the insured or their family will usually suffer a significant financial burden on the death of the bread-winner. Not all of the members in the fund will suffer a loss at any time since the risks of death or disability are independent across the different lives. In any year only a proportion of the lives will die and suffer a loss. If the total losses suffered by the individuals in the fund is averaged across all of the members of the fund then the loss each faces will be the average for all the members of the fund. This average loss will be much less variable than the individual loss.

Similarly in general insurance, a large number of risks such as injury at work, fire, or motor vehicle damage are relatively independent across different risks. By pooling the risks, individuals will be better off since they will replace a large uncertain risk with an average of the total risk for all of the members of the insurance company exposed to similar risks.

Many financial security systems involve the pooling of savings for retirement and other longer term financial needs. These investment funds can also benefit from pooling. Financial markets also allow individuals to minimize risks on their own account if they wish to do so. In the case of investment risks, the risks are not independent since they are influenced by economy wide factors that affect all investments in a similar way. Investment risks can be diversified by holding a portfolio of shares, loans and other investments. This is the strategy of *diversification* or not putting "all your eggs in one basket".

Independent risks

It will first be assumed that individual losses are independent. This means that the probability that one individual suffers a loss is not affected by whether or not another individual has suffered a loss. The risk of dying for most individuals is an independent risk since the chance that a person dies does not usually depend on whether or not someone else has died.

Consider a pool of n independent risks X_i for $i = 1$ to n. Assume that each risk will have a loss of L with probability q or no loss with probability $(1 - q)$. Each risk can incur only one loss.

Thus for each risk

$$\begin{aligned} \Pr(X_i = L) &= q \\ \Pr(X_i = 0) &= (1 - q) \end{aligned}$$

If we consider each individual risk without any pooling then the expected loss is given by

$$\begin{aligned} E[X_i] &= \mu_i \\ &= qL + (1 - q)\,0 \\ &= qL \end{aligned}$$

If we add up all of the losses for a group of independent individual risks and divide the total loss by the number of individuals then this is the average loss. This will be close in value to the expected loss although not necessarily equal to the expected loss. As we increase the number of independent individuals then the average becomes closer to the expected loss.

This is an application of the *law of large numbers.*

Example 7.1 *Consider the distribution of the average number of heads on a toss of two fair coins.*

Solution 7.1 *Toss two coins 10 times and for each toss count the number of heads. Calculate the average number of heads on the two coins by summing the number of heads on each toss and dividing by the number of tosses. Repeat this many times noting the average number of heads each time. Now do this experiment by tossing the two coins 20, 50 and 100 times. You may wish to simulate this experiment on a computer (in which case you can do it 1,000 and 10,000 times or more). What happens as the number of times the coins are tossed increases?*

The variance of each individual risk will be

$$\begin{aligned} var\left[X_i\right] &= \sigma_i^2 \\ &= q\left[L - qL\right]^2 + (1-q)\left[0 - qL\right]^2 \\ &= q\left(1-q\right)^2 L^2 + (1-q)\, q^2 L^2 \\ &= q\left(1-q\right) L^2 \end{aligned}$$

Example 7.2 *An individual has a* 0.01 *chance of dying and will receive \$10,000 on death. Calculate the expected death benefit and variance of the death benefit for this individual.*

Solution 7.2 *The expected benefit is*

$$0.01 \times 10,000 = 100$$

and the variance is

$$0.01 \times 0.99 \times 10,000^2 = 990,000$$

The standard deviation is $\sqrt{990,000} = 995$.

Assume that some individuals pool their risk and each agrees to each pay the average of the total losses into the pool. The pool will then be used to pay the losses of the members of the pool who incur a loss.

Note that both those who do not incur a loss and those who incur a loss will pay the average of the total losses into the pool. Each individual in the pool will replace a large uncertain loss with the average loss when they pool their risks.

The average of the losses for these individuals will be

$$A = \frac{\sum_{i=1}^{n} X_i}{n}$$

with $X_i = L$ if a loss occurs or $X_i = 0$ if no loss occurs.

The **expected value of the average loss** for the pool will be

$$\begin{aligned} E\left[A\right] &= \mu_A \\ &= E\left[\frac{\sum_{i=1}^{n} X_i}{n}\right] \end{aligned}$$

In order to evaluate this we need to determine the probability that j individuals will have a loss for $j = 0$ to n.

If j individuals have a loss then the total loss paid by the pool will be jL since each of the j individuals will incur a loss of L. The following assumptions are made

- each individual risk is independent,
- each individual can have only one loss, and
- the loss is for a fixed amount.

Under these assumptions, the number of losses has a *Binomial* (n, q) distribution so that

$$\Pr\left(\sum_{i=1}^{n} X_i = jL\right) = \binom{n}{j} q^j (1-q)^{n-j} \quad j = 0, 1, 2 \ldots$$

The expected value of the average loss can be evaluated as

$$\begin{aligned}
\mu_A &= E\left[\frac{\sum_{i=1}^{n} X_i}{n}\right] \\
&= \sum_{j=0}^{n} \frac{jL}{n} \binom{n}{j} q^j (1-q)^{n-j} \\
&= \frac{L}{n} \sum_{j=0}^{n} j \binom{n}{j} q^j (1-q)^{n-j} \\
&= \frac{L}{n} nq \\
&= qL
\end{aligned}$$

where the second last line uses the result that the expected value of a *Binomial* (n, q) random variable is nq.

Thus we see that the **expected value of the average loss is the same as the expected value of the individual loss**.

The variance of the average loss will be

$$\begin{aligned}
\sigma_A^2 &= E\left[A - \mu_A\right]^2 \\
&= E\left[A^2 - 2A\mu_A + \mu_A^2\right] \\
&= E\left[A^2\right] - 2E\left[A\right]\mu_A + \mu_A^2 \\
&= E\left[A^2\right] - \mu_A^2
\end{aligned}$$

where

$$A = \frac{\sum_{i=1}^{n} X_i}{n}.$$

Note that deriving this expression for σ_A^2 we have used the following properties of the expected value:

$$E\left[k\right] = k$$

where k is a constant.

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

where X and Y are random variables, that need not be independent, and a, b and c are constants.

We can use these properties to show that for any random variable X we have that

$$Var[X] = E[X^2] - E[X]^2$$

Now consider $E[A^2]$. We have

$$\begin{aligned} E[A^2] &= \sum_{j=0}^{n} \left[\frac{jL}{n}\right]^2 \binom{n}{j} q^j (1-q)^{n-j} \\ &= \left[\frac{L}{n}\right]^2 \sum_{j=0}^{n} j^2 \binom{n}{j} q^j (1-q)^{n-j} \\ &= \left[\frac{L}{n}\right]^2 \left[nq(1-q) + n^2q^2\right] \\ &= \frac{q(1-q)L^2}{n} + q^2L^2 \end{aligned}$$

The second last line uses a result from one of the exercises in Chapter 2.

Recall that it was shown that the variance of a *Binomial* (n, q) random variable is

$$nq(1-q)$$

Since

$$Var[X] = E[X^2] - E[X]^2$$

we have that

$$E[X^2] = Var[X] + E[X]^2$$

Thus for a *Binomial* (n, q) random variable we have that

$$E[X^2] = nq(1-q) + n^2q^2$$

Using the above results we have that

$$\begin{aligned} \sigma_A^2 &= E[A^2] - \mu_A^2 \\ &= \frac{q(1-q)L^2}{n} + q^2L^2 - q^2L^2 \\ &= \frac{q(1-q)L^2}{n} \end{aligned}$$

We see that the variance of the average of the pool losses is the variance of each individual loss divided by the number of individuals in the pool. The larger the number in the pool the less variable the average loss for the members in the pool.

Exercise 7.1 *A group of independent individuals have exposure to a risk causing a loss of \$100,000 with probability of occurring of 0.01. Determine the expected value and variance of the average losses for a group of size 10, 50, 100, 1000, and 10,000.*

So far we have considered the case where the individuals in the pool can only incur a single loss and the amount of the loss is a fixed amount. This case is similar to a life insurance company with policies providing death benefits with a fixed sum assured where individuals can only die once.

In general insurance, individuals can incur more than one loss and the amount of the loss is variable. For example in motor vehicle insurance an individual can have more than one accident in a year and the amount of damage incurred will depend on the seriousness of the accident. This case is more complicated and studied in more detail in later actuarial subjects in the area of insurance risk modelling.

Provided that individuals are risk averse, the expected utility of each individual who participates in the risk pooling will be higher as a result of exchanging the individual risk X_i for the average risk for the total pool of $\frac{\sum_{i=1}^{n} X_i}{n}$. Both have the same expected value. However the average loss for the total pool is less variable.

As noted in the previous Chapter, risk averse individuals prefer certainty to uncertainty. The uncertainty of a risk is sometimes measured by its variance. The lower the variance, for the same expected value, the lower the uncertainty and hence the higher the expected utility.

The benefits from pooling arise from independent risks. The statistical distribution of the risks can vary from the simple examples we have considered so far. The following example illustrates a case with more than one loss.

Example 7.3 *Two independent risks each have the following probability distribution*

Loss	***Probability***
100,000	*0.01*
50,000	*0.05*
0	*0.94*

- *Calculate the expected loss and the variance of the loss for the risks.*
- *Assume that the risks are pooled together. Determine the probability distribution of the average loss, as well as the expected value and the variance of the average loss.*

- *Assume that the individuals have a log utility function with* $v(W) = \ln(1+W)$. *Assume that the individual has current wealth 100,000. Determine the expected utility for an individual before and after pooling the risk.*

Solution 7.3 *Each risk has the same expected value and variance.*

- *The expected value is*

$$\begin{aligned} & 100,000 \times 0.01 + 50,000 \times 0.05 + 0 \times 0.94 \\ = \; & 3,500 \end{aligned}$$

- *The variance is*

$$\begin{aligned} & (100,000 - 3,500)^2 \times 0.01 + (50,000 - 3,500)^2 \times 0.05 + (0 - 3,500)^2 \times 0.94 \\ = \; & 212,750,000 \end{aligned}$$

The standard deviation is $\sqrt{212,750,000} = 14,585.95$.

- *If the risks are pooled then the following table gives each of the possible average losses with their respective probabilities (this is the probability distribution):*

Risk 1 value	***Risk 2 value***	***Average***	***Probability***
100,000	*100,000*	*100,000*	$0.01 \times 0.01 = 0.0001$
100,000	*50,000*	*75,000*	$0.01 \times 0.05 = 0.0005$
100,000	*0*	*50,000*	$0.01 \times 0.94 = 0.0094$
50,000	*100,000*	*75,000*	$0.05 \times 0.01 = 0.0005$
50,000	*50,000*	*50,000*	$0.05 \times 0.05 = 0.0025$
50,000	*0*	*25,000*	$0.05 \times 0.94 = 0.0470$
0	*100,000*	*50,000*	$0.94 \times 0.01 = 0.0094$
0	*50,000*	*25,000*	$0.94 \times 0.05 = 0.0470$
0	*0*	*0*	$0.94 \times 0.94 = 0.8836$

This can be summarized for each distinct average value as follows:

Average	***Probability***
100,000	0.0001
75,000	0.0010
50,000	0.0213
25,000	0.0940
0	0.8836

The expected value of the average loss is

$$\begin{aligned} & \left(\begin{array}{c} 100,000 \times 0.0001 + 75,000 \times 0.0010 + 50,000 \times 0.0213 \\ +25,000 \times 0.0940 + 0 \times 0.8836 \end{array} \right) \\ = \; & 3,500 \end{aligned}$$

The variance of the average loss is

$$\begin{aligned} & \left(\begin{array}{c} [100,000 - 3,500]^2 \times 0.0001 \\ + [75,000 - 3,500]^2 \times 0.0010 \\ + [50,000 - 3,500]^2 \times 0.0213 \\ + [25,000 - 3,500]^2 \times 0.0940 \\ + [0 - 3,500]^2 \times 0.8836 \end{array} \right) \\ = \quad & 106,375,000 \end{aligned}$$

The standard deviation of the average loss is 10,313.83. Note that the variance of the average loss is half the variance of the individual loss (as it should).

- *The wealth after the loss will be 100,000-L where L is the loss. The expected utility before pooling is*

$$\begin{aligned} & \ln(1) \times 0.01 + \ln(50,001) \times 0.05 + \ln(100001) \times 0.94 \\ = \quad & 11.36315 \end{aligned}$$

The expected utility after pooling the risk, where each individual will pay the average cost will be

$$\left[\begin{array}{c} \ln(1) \times 0.0001 + \ln(25,001) \times 0.0010 + \ln(50,001) \times 0.0213 \\ + \ln(75,001) \times 0.0940 + \ln(100001) \times 0.8836 \\ = 11.46859 \end{array} \right]$$

We see that expected utility is higher with risk pooling.

Exercise 7.2 *Consider* **three** *independent risks each with probability distribution*

Loss	***Probability***
100,000	*0.1*
0	*0.9*

Determine the expected utility before and after pooling the risks for an individual with a log utility function with $v(W) = \ln(1 + W)$.

Correlation

In many cases risks are not independent. Life insurance companies sell group products covering all the lives in a particular company or all the members of a superannuation fund who may all be exposed to a common risk. Insurance risks covering damage from catastrophes such as hurricanes or an earthquake will also not be independent. Risks that are located in the same geographical area all will suffer damage if a catastrophe occurs in the area.

Investment market risks are also not independent. When a life insurance company or superannuation fund invests its funds in financial assets the risks involved are linked through economy wide factors that affect all investments to a greater or lesser extent. For example, the value of all investments including shares and loans will all be affected by inflation and interest rates to a greater or lesser extent.

The value of policy liabilities of an insurance company or a superannuation fund will also be influenced by economic factors such as interest rates. Since the difference between the value of the policy liabilities and the value of the assets of an insurance company or superannuation fund determines the value of the equity of the fund, the solvency of an insurance company depends on an understanding of non-independent risks.

In order to understand what happens when risks are not independent it is necessary to introduce the concept of *correlation*.

Correlation is a measure of dependency between random variables. Correlation measures the extent to which a random variable takes values above (or below) its expected value compared with the extent to which another random variable takes values above (or below) its expected value. It measures *linear dependency.*

Some risks are dependent but do not have any linear dependence. For these risks the dependence is non-linear and the correlation will be zero. If two random variables have zero correlation then this does not mean that they are independent. However any two risks that are independent will have zero correlation.

Correlation is a measure that takes a value between -1 and +1. A negative value indicates that the random variables have *negative correlation*. This means that the two risks tend to move in opposite directions. Thus when one random variable takes above average values (below average values) then the other takes values below average (above average).

Example 7.4 *Give an example of random variables in insurance that you might expect to be negatively correlated.*

Solution 7.4 *Employment levels and claims rates in workers compensation often have negative correlation. If employment is below average levels then the claims rate on workers compensation policies may be higher since there is less incentive to return to work in times of low employment.*

If two random variables have positive correlation then they both tend to take above average values or below average values at the same time. They tend to move in the same direction. *Perfect positive correlation* is where the correlation is equal to +1.

Independent risks have zero correlation.

In order to determine the correlation between two random variables it is necessary to measure the *covariance* of the random variables. Covariance of two random variables X and Y is defined as

$$Cov\left(X,Y\right)=E\left[\left(X-E\left[X\right]\right)\left(Y-E\left[Y\right]\right)\right]$$

For two discrete random variables this is determined by

$$\sum_i\sum_j \Pr\left[X=x_i \text{ and } Y=y_j\right]\left(x_i-E\left[X\right]\right)\left(y_j-E\left[Y\right]\right)$$

Recall that if two random losses A and B are independent then

$$\begin{aligned}\Pr\left(A \text{ and } B\right) &= \Pr\left(A|B\right)\Pr\left(B\right)\\ &= \Pr\left(A\right)\Pr\left(B\right)\end{aligned}$$

where $\Pr\left(A|B\right)$ is the conditional probability that A occurs give that B has occurred.

For independence the probability that A occurs does not depend on B so that $\Pr\left(A|B\right)=\Pr\left(A\right)$.

Thus we can just multiply probabilities to determine probabilities that both X and Y take particular values for independent risks. If random variables are not independent then we must specify the probabilities for all possible combinations of the values of X and Y.

In order to measure the covariance of two random variables we require their *joint probability distribution*. The joint probability distribution gives the probabilities for all possible values of the two random variables.

Example 7.5 *The average claims cost for two different insurance Classes in a year depend on whether the economy is booming, normal or in recession.*
The following table gives the average claims cost for these Classes in each state of the economy along with the probability that the economy will be in the state.

State	***Probability of State***	***Average Claims Cost A***	***Average Claims Cost B***
Boom	*0.1*	*1,500*	*500*
Normal	*0.6*	*1,000*	*1,000*
Recession	*0.3*	*500*	*1,500*

The probability is the probability that the average cost will equal the values shown in the corresponding row.

- *Calculate the expected value and variance for each insurance class.*
- *Calculate the covariance between A and B and comment on your answer.*

Solution 7.5 *The expected value for Class A is*

$$\begin{aligned}\mu_A &= 1,500 \times 0.1 + 1,000 \times 0.6 + 500 \times 0.3 \\ &= 900\end{aligned}$$

The expected value for Class B is

$$\begin{aligned}\mu_B &= 500 \times 0.1 + 1,000 \times 0.6 + 1,500 \times 0.3 \\ &= 1,100\end{aligned}$$

The variance for Class A is

$$\begin{aligned}\sigma_A^2 &= (1,500 - 900)^2 \times 0.1 \\ &\quad + (1,000 - 900)^2 \times 0.6 \\ &\quad + (500 - 900)^2 \times 0.3 \\ &= 90,000\end{aligned}$$

or a standard deviation of 300.
The variance for Class B is

$$\begin{aligned}\sigma_B^2 &= (500 - 1,100)^2 \times 0.1 \\ &\quad + (1,000 - 1,100)^2 \times 0.6 \\ &\quad + (1,500 - 1,100)^2 \times 0.3 \\ &= 90,000\end{aligned}$$

or a standard deviation of 300.
The covariance between A and B is

$$\begin{aligned}cov\,(A, B) &= \left[\begin{array}{c} (1,500 - 900)\,(500 - 1,100) \times 0.1 \\ + (1,000 - 900)\,(1,000 - 1,100) \times 0.\bar{6} \\ + (500 - 900)\,(1,500 - 1,100) \times 0.3 \end{array}\right] \\ &= -90,000\end{aligned}$$

Exercise 7.3 *Consider the average claims cost in the previous example. The average claims cost for another class (Class C) is given below.*

State	***Probability of State***	***Average Claims Cost A***	***Average Claims Cost B***	***Average Claims Cost C***
Boom	*0.1*	*1,500*	*500*	*1,000*
Normal	*0.6*	*1,000*	*1,000*	*1,000*
Recession	*0.3*	*500*	*1,500*	*1,000*

- *Calculate the expected value and variance of the average claims cost for Class C.*
- *Calculate the covariance between A and C, and B and C and comment on your answers.*

The correlation between two random variables is defined as

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

where σ_X is the standard deviation of the random variable X and σ_Y is the standard deviation of the random variable Y. Note that we can write the covariance in terms of the correlation coefficient and the standard deviation of each of the risks using the definition of correlation as follows:

$$Cov(X,Y) = \rho(X,Y)\,\sigma_X \sigma_Y$$

Example 7.6 *Use the average claims cost in the previous example and exercise to determine the correlation between risks A and B.*

Solution 7.6 *Correlation between A and B is*

$$\begin{aligned} \rho(A,B) &= \frac{Cov(A,B)}{\sigma_A \sigma_B} \\ &= \frac{-90,000}{300 \times 300} \\ &= -1 \end{aligned}$$

Exercise 7.4 *Determine the correlation between A and C and B and C for the data on average claims cost in the previous exercise.*

To understand what happens when correlated risks are pooled we consider the simplest case of two losses X_1 and X_2 each with the same expected value μ and the same variance σ^2.

The risks have correlation $\rho(X,Y) = \rho$, equal variances with $\sigma_X = \sigma$ and $\sigma_Y = \sigma$ so that the covariance between the two losses is

$$\begin{aligned} & \rho(X,Y)\,\sigma_X \sigma_Y \\ = \; & \rho\sigma^2 \end{aligned}$$

The average loss is

$$\frac{X_1 + X_2}{2}$$

with expected value

$$
\begin{aligned}
& E\left[\frac{X_1+X_2}{2}\right] \\
= & \frac{E\left[X_1+X_2\right]}{2} \\
= & \frac{E\left[X_1\right]+E\left[X_2\right]}{2} \\
= & \frac{2\mu}{2} \\
= & \mu
\end{aligned}
$$

The variance of the average loss will be

$$
\begin{aligned}
& Var\left[\frac{X_1+X_2}{2}\right] \\
= & E\left[\left(\frac{X_1+X_2}{2}-\mu\right)^2\right] \\
= & E\left[\left(\frac{\left[X_1-\mu\right]+\left[X_2-\mu\right]}{2}\right)^2\right] \\
= & \frac{1}{4}E\left[\left[X_1-\mu\right]^2+2\left[X_1-\mu\right]\left[X_2-\mu\right]+\left[X_2-\mu\right]^2\right] \\
= & \frac{1}{4}\left[\sigma^2+2Cov\left(X_1,X_2\right)+\sigma^2\right] \\
= & \frac{\sigma^2+Cov\left(X_1,X_2\right)}{2} \\
= & \frac{\sigma^2}{2}\left(1+\rho\right)
\end{aligned}
$$

If the losses were independent ($\rho=0$) then the variance of the average of the two losses would be $\frac{\sigma^2}{2}$.

We see that pooling of correlated risks multiplies the variance, relative to the variance if the risks were independent, by the factor $(1+\rho)$ where $-1 \leq \rho \leq 1$.

If two risks are positively correlated then this will **reduce** the benefits of pooling of risks.

If $\rho=1$, so that the risks are perfectly positively correlated, then there is no benefit from pooling risks.

However if the correlation is less than 1 and positive then there is benefit from pooling risks.

If risks have negative correlation then it is possible to pool them and reduce the variability below that of independent risks. In fact for the case of perfectly negatively correlated risks it is possible to eliminate all variability by pooling the risks.

These results extend to more than just two risks. The probability required for the case of more than two risks is covered in later subjects in actuarial studies.

Example 7.7 *Consider two risks with*

$$E[A] = E[B] = 20000$$

$$\sigma_A = \sigma_B = 5000$$

If the standard deviation of the pooled risk (the average of the two risks) is 2,500, determine the correlation between the two risks.

Solution 7.7 *We have that*

$$\frac{\sigma^2}{2}(1+\rho) = (2,500)^2$$

where $\sigma^2 = (5000)^2$. *Therefore, the correlation is given by*

$$\begin{aligned} \rho &= 2\frac{(2,500)^2}{(5000)^2} - 1 \\ &= -0.5 \end{aligned}$$

Exercise 7.5 *For the two risks with*

$$E[A] = E[B] = 20000$$

$$\sigma_A = \sigma_B = 5000$$

determine the standard deviation of the pooled risk for values of the correlation between the two risks of -1, –0.75, 0.5, -0.25, 0. +0.25, +0.5, +0.75, and +1.0.

7.4 Risk Classification

The financial soundness of a financial security scheme such as a life insurance company, non life insurance company, superannuation fund or social security system depends on many factors. A key factor is charging a premium for the benefits provided by the scheme that reflects the risks involved in order to ensure fair treatment of the members of the system. This requires the identification of the risk factors that determine the losses for the risks.

Risks with the same expected loss can be charged a rate equal to the average loss when pooled even if the risks are correlated. For risks with different expected losses it is necessary to charge a fair rate when pooled with other risks that reflects their different expected loss.

In order to pool different risks it is necessary to identify risks with different expected losses and to charge a different rate or premium reflecting these differing expected losses. Risks with similar expected losses are referred to as *homogeneous* risk classes.

In order to classify risks into homogeneous classes it is necessary to identify the risk factors that influence the loss.

For life insurance, risk classes are determined using age, sex, smoking status and degree of impairment as risk factors. For each insurance class different survival probabilities would be used to calculate premium rates to allow for the different risk of death or survival. The expected losses for these classes are determined from historical data using statistical analysis of losses for different risk factors.

For motor vehicle insurance the risk factors used include age and sex of the driver, age and make of vehicle, and where the vehicle is garaged.

For workers' compensation insurance, the risk factors will include the industry of the employer. A mining company will have a higher expected loss due to workers' injuries than a bank.

The process of assessing the expected claims cost, determining the fair rate and deciding if the individual can participate in the financial security scheme is called *underwriting.*

Example 7.8 *Explain what is meant by underwriting in life insurance.*

Solution 7.8 *For life insurance, underwriting involves the assessment of the risk using personal medical statements and health examinations in order to decide if the life can be insured at standard premium rates for the age and sex of the insured, or whether the life should be charged an additional premium or declined insurance.*

7.5 Rating and Experience Adjustments

It is not always possible to classify risks into homogeneous groups. This is because the risk can be difficult to accurately assess based on the risk factors for which information is available. In these cases it is possible to take into account the actual experience of the risk in determining the rate actually charged. This is done using *experience rating schemes.*

One of the schemes used in motor vehicle insurance for experience rating is the *No-Claim-Discount* scheme (NCD).

In an NCD scheme a policyholder's premium will depend on whether or not they had a claim in the previous year. Initially they will be charged the full premium for their risk class based on risk factors including age and make of vehicle. The insurance company will allow a policyholder a number of different No Claim Discount levels depending on claims experience.

If a policyholder has a claim free year then they will move up to a higher discount level. If they have one or more claims in a year then they will move to a lower level. The precise rules differ from company to company although most are very similar.

Example 7.9 *An insurance company allows the following NCD levels*

NCD Level	***Discount %***
0	*0*
1	*30*
2	*60*

If a policyholder has no claims in a year then they move to the next higher level of discount (unless they are already on the highest level, in which case they stay there). If a policyholder has one or more claims in a year then they move to the next lowest level of discount (unless they are already on the lowest level, in which case they stay there). The probability of no claims in a year is assumed to be 0.9.

- *Calculate the probability that a new policyholder will be on the maximum NCD level in the third year.*
- *Assuming that the proportions become stable, out of a new group of policyholders what proportion will eventually be paying the full premium?*

Solution 7.9 *To be on the maximum NCD level in the third year the policyholder must have no claims in the first two years of the policy.*
The probability is

$$0.9^2 = 0.81$$

If we let π_i for $i = 0, 1, 2$ to be the long run proportions on NCD level i then we can determine equations for the π_i by noting that when the proportion on each level is stable we will have

$$\pi_0 + \pi_1 + \pi_2 = 1$$

$$\pi_0 = 0.1\,(\pi_0 + \pi_1)$$

$$\pi_1 = 0.9\pi_0 + 0.1\pi_2$$

$$\pi_2 = 0.9\pi_1 + 0.9\pi_2$$

The first equation just states that the sum of the proportions in the different NCD levels must sum to 1.
The second equation says that the proportion in state 0 will equal those from state 0 and state 1 who have one or more claims in the previous year.
The third equation says that the proportion in state 1 will equal those in state 0 who have no claim and those in state 2 who have 1 or more claims in the previous year.
The last equation says that the proportion in state 2 will equal those in state 1 and state 2 who don't have a claim in the previous year.
Rearranging the second equation gives

$$\pi_0 = \frac{0.1}{0.9}\pi_1$$

Rearranging the fourth equation gives

$$\pi_2 = \frac{0.9}{0.1}\pi_1$$

Substituting into the first equation gives

$$\frac{0.1}{0.9}\pi_1 + \pi_1 + \frac{0.9}{0.1}\pi_1 = 1$$

So that

$$\pi_1 = 0.0989011$$

$$\pi_0 = 0.0109890$$

$$\pi_2 = 0.8901099$$

Thus the long run proportion of a new group of policyholders paying the full premium will be $\pi_0 = 0.0109890$ *or about 1%.*

The NCD scheme is an example of a Markov chain. A Markov chain consists of a number of states with probabilities that the chain will move from one state to another over a time interval.
The probability that the chain moves from state i to state j over a single time interval will be denoted by p_{ij}. The *transition probability matrix* is defined as

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{bmatrix}$$

This matrix gives the probabilities of moving from each and every state at the current time to each and every state at the next time. It is possible to show that for the NCD scheme the probability of being in state j at time n given that the chain starts in state i at time 0 is given by the ij^{th} element of the matrix P^n (the matrix P raised to the power of n). This is easily demonstrated by example.

Example 7.10 *For the NCD scheme in the previous example, write down the transition probability matrix. Determine the probabilities of being in each of the states after 2, 5, 10, 20 and 25 years.*

Solution 7.10 *We have*

$$P = \begin{bmatrix} 0.1 & 0.9 & 0 \\ 0.1 & 0 & 0.9 \\ 0 & 0.1 & 0.9 \end{bmatrix}$$

We can calculate the powers of this matrix using a numerical package such as Maple, Matlab or using the matrix manipulation commands in Excel. We obtain

$$P^2 = \begin{bmatrix} 0.1 & 0.09 & 0.81 \\ 0.01 & 0.18 & 0.81 \\ 0.01 & 0.09 & 0.9 \end{bmatrix}$$

It can easily be verified that the ij^{th} element of the matrix P^2 is the probability that the level of discount is j at time 2 given that it as level i at time 1.
Continuing with the matrix calculations we obtain

$$P^5 = \begin{bmatrix} 0.01171 & 0.10539 & 0.8829 \\ 0.01171 & 0.0981 & 0.89019 \\ 0.0109 & 0.09891 & 0.89019 \end{bmatrix}$$

$$P^{10} = \begin{bmatrix} 0.0109949 & 0.0989005 & 0.890105 \\ 0.0109889 & 0.0989064 & 0.890105 \\ 0.0109889 & 0.0989005 & 0.89011 \end{bmatrix}$$

$$P^{20} = \begin{bmatrix} 0.010989 & 0.0989011 & 0.89011 \\ 0.010989 & 0.0989011 & 0.89011 \\ 0.010989 & 0.0989011 & 0.89011 \end{bmatrix}$$

$$P^{25} = \begin{bmatrix} 0.010989 & 0.0989011 & 0.89011 \\ 0.010989 & 0.0989011 & 0.89011 \\ 0.010989 & 0.0989011 & 0.89011 \end{bmatrix}$$

We notice that the matrix probabilities become stable. In fact after only 15 time periods the matrix probabilities are stable.

Exercise 7.6 *An NCD scheme has three levels of discount. The rules for moving between the levels are that if no claims occur then the policyholder moves to the next higher level. If a single claim occurs then the policyholder moves to the next lowest level. If two or more claims occur then the policyholder moves back two levels. Denote the probability of no claims by q_0 and the probability of a single claim by q_1.*

- *Show that the transition probability matrix is*

$$P = \begin{bmatrix} 1-q_0 & q_0 & 0 \\ 1-q_0 & 0 & q_0 \\ 1-q_0-q_1 & q_1 & q_0 \end{bmatrix}$$

- *Also show that the long run proportions on each level will be*

$$\pi_0 = \frac{(1-q_0-q_0q_1)}{(1-q_0q_1)}$$

$$\pi_1 = \frac{q_0\,(1-q_0)}{(1-q_0q_1)}$$

$$\pi_2 = \frac{q_0^2}{(1-q_0q_1)}$$

- *Assume that claim numbers have a Poisson(λ) probability density. Determine the long run proportions in each NCD level for $\lambda = 0.1, 0.2, 0.3, 0.4.$ and 0.5.*

- *Comment on how well the NCD scheme classifies different risks.*

Markov chains are studied in more detail in later actuarial subjects.

7.6 Moral Hazard and Adverse selection

Moral hazard occurs where an individual has an incentive not to reduce risk because they are covered by the financial security scheme. Individuals with insurance have less incentive to prevent losses than those without. This is because, if they have insurance and a loss occurs, then the loss is met by the insurance company. If they try to prevent or reduce losses then the main beneficiary will be the insurance company.

Moral hazard only occurs where the insured can influence the losses on an insurance policy. If they can not influence the losses by their actions then moral hazard is not a problem. Experience rating can be used to control moral hazard. By altering the premium in line with actual claims experience the insured will bear some of the cost of adverse claims experience. They will then have an incentive to reduce risks.

Exercise 7.7 *WeatherWise is a US company that guarantees the total cost of a consumer's energy supply regardless of weather conditions. WeatherWise uses a year of data on the consumer's actual energy costs including both summer and winter to assess the guaranteed bill amount. If the consumer's energy bill is higher than the guaranteed bill then WeatherWise pays the difference. If the energy bill is lower then the consumer pays Weatherwise the difference. What is the moral hazard with this product?*

Thus insurance companies try to limit moral hazard by offering insurance policies that share the risk with the insured. As well as experience rating schemes, insurance companies use a *deductible or an excess* where the policyholder has to pay the first part of each claim. This gives the policyholder an incentive to prevent loss.

Example 7.11 *Consider a motor vehicle policy with a deductible of $500. Describe the payments made if a claim occurs.*

Solution 7.11 *In this case the insurance company will pay the claims cost if the claim amount exceeds $500 less the first $500 of the claim. All claims less than $500 are met by the policyholder. If a claim exceeds $500 then the insurance company pays the claim amount less $500.*

Adverse selection occurs where a financial security system does not, or is unable to, distinguish between different risks and hence does not charge a different rate for the differing risks. If membership is voluntary and individuals can assess their own risk then those with higher risk will have incentives to participate in the financial security scheme if the rate charged does not reflect the higher risk. Adverse selection relies on individuals having more information about their risk than those offering the financial security products. This is referred to as *asymmetric information.* It occurs where the risk classification system results in non-homogeneous risk classes.

Risk classification aims to reduce adverse selection. Where risks can not be assessed and classified then premiums will tend to be higher to reflect adverse selection. Thus better risks will tend to be denied coverage at a fair rate. If the level of deductible can be selected by the individual then the better risks will tend to select higher deductibles. This is one way of mitigating the effect of adverse selection.

Community rating is a financial security system where the same rate is charged to all members of the scheme. Financial security schemes can be *mandatory* or *voluntary.* Mandatory schemes require membership of the system by law. Most community rating schemes are mandatory. This is to control adverse selection. Often publicly provided financial security benefits are mandatory. The cost of mandatory public schemes is usually met through the tax system or through a separate levy that is collected as part of the tax system. The social security scheme in Australia is funded through the taxation system. The health care system is in part funded by the Medicare levy which is also collected as part of the tax system.

Voluntary schemes have optional membership. In Australia the private provision of retirement benefits for employees is mandatory. Employers must contribute a specified percentage of employee salary, currently 9%, into a superannuation scheme.

7.7 Actuarial Soundness and Ruin

The likelihood that a financial security scheme will be able to meet its future obligations reflects the degree of *actuarial soundness* of the system. Actuaries assess the probability of ruin in order to quantify the level of actuarial soundness. In order to do this they simulate the operation of the scheme into the future using sophisticated financial models.

The financial models are used to assess the proportion of occasions in the future when the scheme does not have sufficient assets to cover the payment of the future liabilities. These are the occasions when the scheme is ruined. The model results can be used to recommend actions to be taken to reduce the ruin probability. These could include increasing premium rates, altering the investment policy of the fund, altering any reinsurance arrangements or even ceasing to operate.

The key roles that actuaries play in ensuring the soundness of these financial security schemes is in the setting of fair premiums that include adequate loadings for expenses and risk as well as the establishment of an investment policy that takes into account the liabilities of the scheme. Actuaries periodically assess the value of the liabilities of the scheme in order to compare these with the value of the assets.

7.8 Conclusions

This chapter has covered the key concepts underlying risk management of financial security systems. These include pooling of risks, risk classification and charging a fair premium for the risk. The actuarial soundness of the system is assessed periodically to ensure there are sufficient assets to meet the future benefits.

Further detail on insurance risk management can be found in Harrington and Niehaus (1999) ([6]). The statistical basis for insurance is covered in Hossack, Pollard and Zehnwirth (1999) ([9]) including more on experience rating schemes.

7.9 Solutions to Exercises

Ex 7.1 *The expected value for the average loss is the same for each individual loss. This is*

$$\begin{aligned} qL &= 0.01 \times 100,000 \\ &= 1,000 \end{aligned}$$

The variance of the average losses is given by

$$\frac{q\,(1-q)\,L^2}{n}$$

The table below gives the variance (and standard deviation) for groups of size 10, 50, 100, 1000, and 10,000

Number	Variance of Average Loss	Standard Deviation of Average Loss
10.00	9,900,000.00	3,146.43
50.00	1,980,000.00	1,407.12
100.00	990,000.00	994.99
1,000.00	99,000.00	314.64

.

Ex 7.2 *Before pooling the risks the individuals are exposed to the risk*

Loss	***Probability***
100,000	*0.1*
0	*0.9*

The expected utility is calculated in the following table

Loss	Probability	Utility ln(1+W)	Utility ln(1+100000-W)
100,000	0.1	11.51	0.00
0	0.9	0.00	11.51
	Expected Utility	1.15	10.36

In the table the right hand column shows the expected utility for the loss for an individual with wealth of \$100,000. When the risks are pooled the following table gives the probability distribution of the average loss and the expected utility

Loss 1	Loss 2	Loss 3	Average Loss	Probability	Utility ln(1+W)	Utility ln(1+100000-W)
100,000	100,000	100,000	100,000.00	0.001	11.51	0.00
0	100,000	100,000	66,666.67	0.009	11.11	10.41
100,000	0	100,000	66,666.67	0.009	11.11	10.41
0	0	100,000	33,333.33	0.081	10.41	11.11
100,000	100,000	0	66,666.67	0.009	11.11	10.41
0	100,000	0	33,333.33	0.081	10.41	11.11
100,000	0	0	33,333.33	0.081	10.41	11.11
0	0	0	0.00	0.729	0.00	11.51
				Expected Utility	2.84	11.37

The expected utility is higher for the pooled risks which means that if the individuals are risk averse (with a log utility function in this case) then they will prefer to pool the risk.

Ex 7.3 *The table of probabilities is*

State	***Probability of State***	***Average Claims Cost A***	***Average Claims Cost B***	***Average Claims Cost C***
Boom	*0.1*	*1,500*	*500*	*1,000*
Normal	*0.6*	*1,000*	*1,000*	*1,000*
Recesion	*0.3*	*500*	*1,500*	*1,000*

Exercise 7.8 • *The expected value for Class C is \$1000. Note that it is constant regardless of economic conditions. Its variance is zero (it does not vary since it is a constant).*

- *The covariance between A and C, and B and C is calculated in the following table*

State	Probability	A	B	C	(A-muA)*(C-muC)	(B-muB)(C-muC)
Boom	0.1	1,500	500	1,000	0	0
Normal	0.6	1,000	1,000	1,000	0	0
Recession	0.3	500	1,500	1,000	0	0
Expected value		900	1,100	1,000	0	0

Note that the covariance between the constant average claims cost of Class C and the other Classes is zero. It can be shown that the covariance of a constant with another random variable is zero.

Ex 7.4 *The correlation between A and C and B and C are both zero since the covariance is zero.*

Ex 7.5 *The standard deviation of the pooled risk is the square root of the variance. The variance is given by*

$$\frac{\sigma^2}{2}(1+\rho)$$

We have

$$\sigma^2 = (5000)^2 = 25,000,000$$

The table below gives the variance and standard deviation for the risks pooled together

Correlation	Variance	Standard Deviation
-1	0	0.00
-0.75	3,125,000	1767.77
-0.5	6,250,000	2500.00
-0.25	9,375,000	3061.86
0	12,500,000	3535.53
0.25	15,625,000	3952.85
0.5	18,750,000	4330.13
0.75	21,875,000	4677.07
1	25,000,000	5000.00

Ex 7.6 *The ij^{th} element of the transition probability matrix gives the probability that a policyholder currently on level i will move to level j. Consider the lowest level. The chance that a policyholder will stay at the lowest level is the probability that they have one or more claims i.e p_{11} =1 minus the probability of on claims $= 1 - q_0$. The probability that they move up to the next highest level is the probability that they have no claims i.e $p_{12} = q_0$. Since they can not jump two levels, this probability is zero i.e. $p_{13} = 0$.*
Similar arguments apply for the middle level . Now since the policyholder on the middle level will either move up (no claims) or move down a level (one or more claims) we have $p_{21} = 1 - q_0$, $p_{22} = 0$ and $p_{23} = q_0$.
Finally for a policyholder on the third level if they have no claims then they will stay in the highest level, if they have one claim then they will move down one level otherwise if they have two or more claims they will move to the lowest level. We have $p_{31} = 1 - q_0 - q_1$, $p_{32} = q_1$, and $p_{33} = q_0$. Hence we get the transition matrix

$$P = \begin{bmatrix} 1-q_0 & q_0 & 0 \\ 1-q_0 & 0 & q_0 \\ 1-q_0-q_1 & q_1 & q_0 \end{bmatrix}$$

Exercise 7.9

- *The long run proportions on each of the level will be given by*

$$\pi_0 = (1-q_0)\,\pi_0 + (1-q_0)\,\pi_1 + (1-q_0-q_1)\,\pi_2$$

$$\pi_1 = q_0\pi_0 + q_1\pi_2$$

$$\pi_2 = q_0\pi_1 + q_0\pi_2$$

$$\pi_0 + \pi_1 + \pi_2 = 1$$

From the third equation

$$\pi_2 = \frac{q_0}{(1-q_0)}\pi_1$$

From the second equation

$$\pi_1 = q_0\pi_0 + \frac{q_1 q_0}{(1-q_0)}\pi_1$$

and rearranging gives

$$\begin{aligned} \pi_0 &= \frac{1}{q_0}\left[1 - \frac{q_1 q_0}{(1-q_0)}\right]\pi_1 \\ &= \frac{(1-q_0) - q_1 q_0}{q_0\,(1-q_0)}\pi_1 \end{aligned}$$

The last equation then gives

$$\frac{(1-q_0)-q_1q_0}{q_0(1-q_0)}\pi_1+\pi_1+\frac{q_0}{(1-q_0)}\pi_1=1$$

So that

$$\pi_1=\frac{q_0(1-q_0)}{(1-q_0q_1)}$$

We then have

$$\begin{aligned}\pi_2 &= \frac{q_0}{(1-q_0)}\frac{q_0(1-q_0)}{(1-q_0q_1)}\\ &= \frac{q_0^2}{(1-q_0q_1)}\end{aligned}$$

Finally

$$\begin{aligned}\pi_0 &= \frac{(1-q_0)-q_1q_0}{q_0(1-q_0)}\pi_1\\ &= \frac{(1-q_0)-q_1q_0}{q_0(1-q_0)}\frac{q_0(1-q_0)}{(1-q_0q_1)}\\ &= \frac{1-q_0-q_1q_0}{(1-q_0q_1)}\end{aligned}$$

- *The proportions on each level are shown in the table below. For the Poisson*(λ) *probability density we have*

$$q_0=e^{-\lambda}$$

and

$$q_1=\lambda e^{-\lambda}$$

Long Run Proportion on Each Discount Level					
	Poisson parameter				
Discount level	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
Low	0.01447	0.05451	0.11317	0.18280	0.25676
Middle	0.09379	0.17139	0.22985	0.26941	0.29244
High	0.89174	0.77410	0.65698	0.54778	0.45080

- *The proportion in the remaining in the highest discount level is quite high even for fairly high levels of claims rate. For policyholders with a claim rate of* $\lambda=0.5$ *(i.e. a claim every two years on average) there is still 45% on the highest level.*

Ex 7.7 *The moral hazard with this product is that the individual can use more energy than they would otherwise use once they have the contract. They have no incentive to save energy since any savings only benefit WeatherWise. One way around this would be to share any excess differences between the guaranteed and the actual costs with the consumer. This would give them an incentive to save energy.*

Chapter 8
LIFE INSURANCE

8.1 Learning Objectives

The main objectives of this chapter are:

- to overview the basis for the early insurance contracts including assessmentism and scientific life insurance
- to outline the main features of traditional life insurance products, unit-linked and other savings products, and
- to introduce the main techniques of actuarial management of life insurance products including setting premium rates and valuation of policy liabilities.

8.2 Historical Development

8.2.1 Assessmentism

The early life insurance contracts were based on the principle of *assessmentism*. These contracts were short-term and provided coverage for a year at a time. If the life died during the year then the policy paid the sum insured. The premium, paid at the start of the year, was based on the expected death claims risk during the year plus an allowance for expenses.

It was necessary to satisfy a health requirement in order to purchase the insurance. The policy was renewed at the end of the year at the option of the insurance company and the insured. If the insured was in poor health then it was unlikely that the insurance company would renew the policy for another year.

Because mortality rates increased with age, the expected death claims risk increased every year that the policy was renewed. At the older ages it was more likely that the insured would be declined insurance based on their health. If they were in good health then the premium was high and often the insurance could not be afforded.

These early contracts were inadequate since, just at the time that insurance coverage was most needed, it was either not available, because of the poor health of the life insured, or too expensive, because of the high age of the insured.

Example 8.1 *Determine the expected cost of the death risk for a $100,000 sum insured at five year age intervals using the IA64-70 mortality rates based on Australian insured lives.*

Solution 8.1 *The results are shown in the table*

IA64-70 Ultimate (mainly male insured lives)		
Sum Insured		$100,000.00
Age	**qx**	**Death Risk**
20	0.00192	192
25	0.00107	107
30	0.00103	103
35	0.00123	123
40	0.00174	174
45	0.00282	282
50	0.00489	489
55	0.00856	856
60	0.01477	1477
65	0.02491	2491
70	0.04096	4096
75	0.06561	6561
80	0.10221	10221
85	0.15440	15440
90	0.22501	22501
95	0.31445	31445
100	0.41896	41896
105	0.53036	53036
110	0.63817	63817

Notice how the cost of the death risk increases rapidly at the high ages.

In order to remedy these problems a later modification to the assessmentism contracts was to introduce a *guaranteed renewal option* in the insurance policy. This option guaranteed the policyholder the right to renew the policy regardless of health. It did not remove the problem of an increasing premium that made the insurance unaffordable at older ages.

There was also a problem with adverse selection for these guaranteed contracts. The policyholders who did not renew their policies tended to be the lives in better health. Those in worse health tended to take up the option of renewal. When lives with better health withdraw from an insurance contract, this is called *selective withdrawal.* It is an example of adverse selection in insurance.

8.2.2 Scientific Life Insurance

In order to overcome the problems with the assessmentism contracts, James Dodson, a mathematician, developed the principles of *Scientific Life Insurance.* The principles were that

- Health requirements were only required on entry
- Policies were guaranteed renewal regardless of health
- Contracts were for longer terms than one year including for the whole of life
- Premiums charged were level and varied with age at entry and the term of the contract
- Extra premiums were charged for extra health risks.

For these longer term contracts, level premiums were charged for a risk that normally increased with age. The early premiums were higher than the expected death risk and the later premiums were lower than the expected death risk.

In 1762 the Equitable Society was founded on James Dodson's principles of scientific life insurance. This Society was the first insurance company to use the title "actuary". Edward Rowe Mores, MA FSA (Fellow of the Society of Antiquaries), named the position of Chief Executive of the insurance company "Actuary" after the Latin "actuarius".

In the early days life insurance was sometimes used for gambling on lives. This led to the 1774 Life Insurance Act, also called the "Gambling Act", which required *insurable interest* in the life insured by the owner of the policy. The owner of a policy on a life had to have a financial interest in the survival of the life to be insured at the time of sale of the policy.

In 1848 The Institute of Actuaries was formed followed by the Faculty of Actuaries in 1856.

8.2.3 Early Developments in Australia

The early life insurance companies were British life offices. The first life insurance proposal was in 1833. In 1848 the AMP was established as a mutual society. National Mutual (NMLA), now AXA, was established in 1869. Colonial Mutual was established in 1874.

Life insurance companies in Australia were often innovative in product development. For example, NMLA was one of the first life insurance companies in the world to include a *non-forfeiture provision* in its life insurance contracts. When a policyholder does not pay a premium the policy will normally not continue in force and the policy benefits will be forfeited. The non-forfeiture provision automatically kept the policy in force by effectively loaning the premium due from the value of the policy. If the life died then the loaned premiums along with any interest were deducted from the sum insured before being paid to the policyholder.

In the 1880's there was an "invasion" by American life insurance companies. These companies introduced tontine policies. These policies paid higher benefits to the lives that survived. A tontine is an arrangement where the last member of a fund receives all the money remaining in the fund.

By 1935 sickness and income insurance had been introduced by the life insurance companies. Sickness benefits were usually included as a "rider" to a main policy. A rider is an additional benefit to the main life insurance policy. If the life became permanently disabled then the policy would pay the sum insured. Income insurance paid a benefit as a percentage of salary while the insured was unable to work due to sickness.

The Commonwealth government has power to legislate on insurance matters under the Constitution. In 1945 the Life Insurance Act was enacted. This Act governed the operation of life insurance companies until 1995 when the latest Life Insurance Act was introduced.

8.3 Life Insurance Products

In Australia and other developed countries the types of life insurance policy sold at the present time are quite different to the policies these companies developed and sold through the 1800's and early 1900's.

The original policies that these companies sold are referred to as *traditional policies*. Traditional policies are the whole of life and endowment policies. They have been used since the earliest days of scientific life insurance to provide protection in the event of death and for long term savings. Life insurance companies in Australia no longer sell the traditional policies although they still have a significant number of these policies in force.

The traditional whole of life and endowment insurance policies are used in Asian countries.

8.3.1 Whole of life

A whole of life policy pays the sum insured on death of the life insured regardless of when the life dies. Although premiums can be payable as long as the life is alive, the premium payments are usually limited to a specified age, such as to age 75, or for a specified term, such as 40 years. By limiting the premium payment term, the insurance company avoids the situation where lives that survive to advanced ages pay premiums in excess of the sum insured payable on death.

Policies can be either *non-participating* or *participating*. Non-participating policies pay a fixed sum insured on death. The fixed amount is guaranteed as long as the contractual premium is paid. Participating policies share in the profits from the insurance business.

In Australia the profit is credited to policyholders by increasing the sum insured payable on death with a *bonus*. The bonus is usually added as a percentage of the sum insured (*simple bonus*) or as a percentage of the sum insured plus previous bonuses (*compound bonus*). It is called a *reversionary bonus* because it is paid in the future. A *reversion* is a benefit that is paid on the death of a life. Hence the use of the term reversionary bonus.

In North America, profits are paid to policyholders as a dividend which can be used to reduce the premium payable or may be paid as cash.

Example 8.2 *Describe the main features of a participating whole-of-life insurance policy.*

Solution 8.2 *A participating whole-of-life policy with premiums limited to age 65 and sum insured of $100,000 will pay the sum insured plus any bonuses declared on the policy on death whenever death occurs. Level premiums are payable, as long as the life is alive, until age 65 when they cease. Bonuses are added to the sum insured as a percentage. If the bonus was a simple bonus and the rate declared in a year was 2% (20 per thousand of sum insured) then the bonus would increase the sum insured payable on death by $2000.*

8.3.2 Endowment

An endowment assurance pays the sum insured on death within a specified term. At the end of the term of the endowment, if the life has survived then the sum insured is paid. Thus the policy pays the sum insured on death within the term of the insurance or on survival to the end of the term of the policy. The endowment policy combines life insurance cover payable on death (*death cover*) with *savings* payable on maturity of the contract.

Example 8.3 *Describe the main features of a non-participating endowment assurance.*

Solution 8.3 *A 20-year non-participating endowment assurance on a life aged 40 for a sum insured of $100,000 will pay the sum insured if the life dies before age 60 otherwise it will pay the sum insured on maturity at age 60. The policy is non-participating so bonuses are not added to the sum insured.*

A pure endowment pays the sum insured benefit on survival to the end of the term of the policy. With pure endowments the policy does not normally pay any benefit on death prior to maturity. This is unsatisfactory since the policyholder will have paid premiums prior to death yet will receive no benefit.

8.3.3 Term insurance

Term insurance policies pay a sum insured on death within a specified term. They pay nothing on survival to the end of the policy term. This could be for a fixed term such as 10, 20 or 40 years or could be to a particular age, for example to age 65. They guarantee renewal of the policy at the original level premium throughout the term of the policy regardless of health. These policies are referred to as protection or "risk" products since they provide only protection against the risk of death.

There are a variety of different types of term insurance. *Reducing term insurance* has a sum insured that reduces by level amounts during the term of the contract. The reductions in sum insured can be designed to correspond to the balance outstanding under a loan.

A very popular form of term insurance is the *renewable term* insurance policy. These policies allow renewal up to a specified age such as to age 65. The policies are often renewed every year although they could be renewable every five years or for some other frequency. The premium charged is the term insurance premium for the renewal period for the age of the insured at the time of renewal. Thus the premium increases when the policy is renewed. These policies are similar to the early contracts of assessmentism with guaranteed renewal.

8.3.4 Annuities

Life insurance companies sell life annuities. These annuities pay a regular payment provided the life is alive. The annuity ceases on death. The policyholder pays a lump sum, called the *consideration* or the *single premium*, to purchase the annuity.

The payments on an *immediate annuity* commence on payment of the consideration. The standard life annuity pays a level payment as long as the life is alive. An immediate annuity would be purchased by someone retiring at age 65. An immediate annuity purchased with a lump sum retirement benefit will provide a pension payment in retirement.

Deferred annuities commence payment on a future date. A deferred annuity, purchased by someone while they were working, would commence payment on retirement at age 65, for example.

Annuities sold by life insurance companies offer various features including guaranteed payment periods. For example, if an annuity has a guaranteed payment period of 5 years then on death within the first 5 years after commencement of the annuity the purchaser would receive the balance of the first 5 years annuity payments.

The payments on regular annuities are level and fixed. A range of annuities provide for variable annuity payments.

For an *indexed annuity,* the payments are indexed to inflation. Usually the payment is indexed to the Consumer Price Index (CPI) although some are indexed to average weekly earnings. The amount paid on the annuity increases with the rate of inflation.

The payments on a *variable annuity* are linked to the returns on investments. If the annuity payments were linked to share market returns then higher payments would be made when share market returns were high. Payments would fall if the value of the sharemarket fell. The annuity payment could be linked to the returns on different classes of investments not just shares.

Payments on *joint life annuities* depend on the survival or more than one life. For example a joint life annuity could be purchased for a payment as long as least one of two lives are alive. A married couple might purchase an annuity that pays a full payment while both are alive and then 60% of the full annuity payment after the first death of the husband or wife, with the payments ceasing on the death of the last survivor.

8.3.5 Unbundled Policies

In the 1980's the life insurance industry in many developed countries introduced new forms of life insurance product. These products are often referred to as "unbundled" policies since the death and disability cover is separated from the savings component of the traditional policies. There are various types of these products available.

Premiums can be paid as *single premiums* or as *regular premiums*. Single premium contracts have a single up front premium payment at the commencement of the contract. Regular premium contracts pay premiums during the term of the contract, usually in advance.

There are two main types of unbundled contract in Australia. They are the *investment account policy* and the *unit linked policy*. Both these types of policy separate the *savings and protection (insurance) components* of the policy.

These policies operate like bank accounts. An account is established for each policy. The premiums are paid into the account. The expenses are charged to the account including a mortality charge for any life insurance cover and a charge for any disability cover. The balance of the account after expenses and mortality charges is invested by the company.

For an *investment account policy*, the life insurance company invests the account balances into a pooled fund. The allocation of the fund to different types of assets is determined by the life insurance company. At the end of each year the investment return on the pooled fund is credited to the individual policies as an interest credit similar to how interest is credited to a bank account. The rate of interest added to the account is the *crediting rate*. The investment return does not usually include unrealised gains and losses. The insurance company often guarantees an interest rate that will be credited for the following year.

The value of the policy is the amount in the investment account. For some products the initial expenses are not all charged to the account at the start of the policy. Instead the charge for these is spread over the term of the policy. In this case any initial expenses that have not been charged if the policy terminates early will have to be charged as a lump sum amount at termination. This charge is usually called a *surrender* charge since early termination of an insurance policy, when there is a balance in the investment account, is called a surrender.

Unit linked policies operate in a similar way to investment account policies except that the balance of the account is determined by the value of the investments. The premium less expenses and mortality charges is used to purchase a number of units in a pooled fund. Each unit has a value equal to the market value of the assets, including unrealised gains and losses, divided by the number of units owned by all of the policyholders. The policyholder selects the proportion of their premium to be invested in the different asset classes. Thus they can select to have their funds invested in a mix of different types of assets - such as domestic equities, international equities, fixed interest and cash. As the market value of the assets changes then so does the value of a unit in the fund. If these policies surrender then there may also be a surrender charge if all of the initial expenses have not been charged to the policy.

8.3.6 Universal Life

In North America the unbundled life insurance policies are called *universal life insurance.* These are flexible policies that allow the premium and the life insurance cover to be varied by the policyholder. The policyholder nominates a premium and an amount of insurance cover. The premium is used to pay expenses and the mortality charges and is then invested in a range of different asset classes as selected by the insured.

8.3.7 Disability

Disability policies are issued by life insurance companies as additional benefits with long term insurance policies and as separate policies. The additional benefits are called *rider disability benefits* or *TPD (Total and Permanent Disability) benefits.* These benefits pay the sum insured if the life is totally and permanently disabled. This is usually a very severe form of disability where the insured is unable to carry out their own occupation or any occupation.

Separate *disability income policies* provide a percentage of the insured's income in the event of disability for a fixed period or to a fixed age. The definition of disability can vary from company to company. They can be for a disability that prevents the insured carrying out their own occupation or their own or any occupation.

8.3.8 Critical Illness

Critical illness policies pay a benefit if the insured suffers specified illnesses or surgical procedures. For example these policies usually cover cancer, heart attack, stroke, and coronary bypass. Critical illness benefits are often provided along with life insurance cover. The benefit on critical illness is usually the payment of the sum insured that would normally be paid on death under the insurance policy. There are also stand alone critical illness policies.

Exercise 8.1 *Obtain the application forms for a term insurance policy and a sickness insurance policy. You may find some of this information on the WWW. What coverage does the policy provide? What information does the company require for the life to be insured (i.e. health, occupation, age, sex, smoking status etc.)? What premium rates apply for standard lives - how do they vary by age and sex?*

8.3.9 Main Features of Life Insurance Contracts

Life insurance contracts are for long terms. They involve a guarantee to make payments many years in the future. Whole of life insurance and annuities involve payments depending on death or survival for as long as the oldest age of the life table. For example an annuity could involve a commitment to make payments for over 40 years. Term insurance contracts often provide cover for periods as long as twenty or thirty years. Savings products usually have terms of at least 10 years.

The payments are mostly fixed or linked to an index such as the CPI. The amount of the payment can be estimated with a high degree of confidence. For a large number of lives insured, the expected claims payments on death or survival can be estimated. Mortality improvements can cause some uncertainty in the estimate of future payments but an allowance can usually be made for this.

Even though the policies are issued for long terms, in many cases the policies are *lapsed or surrendered* prior to the maturity date of the policy. A policy lapses when a policyholder does not pay the premium and no contractual payment is owed to the policyholder.

A policyholder can surrender their policy prior to the maturity date of the policy and receive a payment on the policy. This payment is called the *surrender value*. Often the surrender value is guaranteed. A minimum surrender value is usually required by life insurance law in many countries.

Lapses and surrenders are hard to forecast because they can depend on future economic conditions. For example if there is a recession then often policyholders with savings policies will need to surrender the policies to access their savings to cover short term needs.

8.4 Actuarial Management

The most important risk and financial management roles of the actuary in life insurance are:

- Design of insurance contracts and rating of risks to ensure that the appropriate rate is charged for participating in the insurance risk pool
- Valuation of policy liabilities to ensure that the profit from the insurance business is realistic and that the company is solvent, and
- Formulation of an investment policy for the assets of the life insurance company.

We will consider a simple term insurance contract as an illustration.

8.4.1 Premium Rating and Valuation of Policy Liability

Premium rating

Consider a term insurance on a life (x) for a term of n years that pays the sum insured S if the life dies during the term and pays nothing if the life survives the term.

To calculate the premium for this policy we will use the *principle of equivalence* where we equate the expected present value of the revenue to the expected present value of the expenses including claims payments. The expected values may be calculated using modified probabilities of death in order to include a risk loading for the insurance company or this could be allowed for by adding a separate profit loading.

Assume that

- the death benefit of S is paid at the end of the year of death

- interest rates are constant and do not vary with the time to receipt of a cash flow, and
- the effective interest rate per year is i.

Claim Payments Recall that we used $T(x)$ *to denote the future lifetime* (a continuous random variable) for a life (x). Thus if X *is the age-at-death* (also a continuous random variable) then the future lifetime is the age at death less the current age:

$$T(x) = X - x$$

For the term insurance the benefit is assumed to be paid at the end of the year of death so that if death occurs in the first year of the policy, when the life is aged x, then the present value of the death benefit will be

$$\frac{S}{(1+i)} = Sv$$

If death occurs when the life is aged $x+1$ then the present value of the benefit will be

$$\frac{S}{(1+i)^2} = Sv^2$$

In general, if death occurs when the life is aged $x+k$ last birthday where $k = 0, 1, 2, \ldots n-1$ then the *present value of the claim payment* at the end of the year of death will be

$$= \begin{cases} Sv^{k+1} & \text{for } k = 0, 1, 2, \ldots n-1 \\ 0 & \text{for } k \geq n \end{cases}$$

In order to determine the premium we need to calculate the **expected present value of the claims payment**. To do this we require the probability that death will occur when the life is aged $x+k$ last birthday where $k = 0, 1, 2, \ldots n-1$.

Let

$$K = \text{ integer part of } T(x)$$

then the probability distribution of K (a discrete random variable) can be readily determined as follows

$$\begin{aligned} \Pr[K = k] &= \Pr[k \leq T(x) < k+1] \\ &= \Pr[x+k \leq X < x+k+1 | X > x] \\ &= {}_{k|}q_x \\ &= {}_kp_x q_{x+k} \end{aligned}$$

This just states that the chance that a life currently aged x will die aged $x+k$ last birthday is the probability that the life survives to age $x+k$ (probability ${}_kp_x$) times the probability that the life dies in the following year aged $x+k$ (probability q_{x+k}).

Example 8.4 *Derive an expression for* $\Pr[K = k]$ *in terms of the survival function* $s(x)$.

Solution 8.4 *Since* $s(x) = \Pr(X > x)$ *we have*

$$\begin{aligned} \Pr[K = k] &= \Pr[x + k \leq X < x + k + 1 | X > x] \\ &= \frac{s(x+k) - s(x+k+1)}{s(x)} \end{aligned}$$

Exercise 8.2 *Show that, using the notation of the life table,*

$$\Pr[K = k] = \frac{d_{x+k}}{l_x}$$

The *expected present value of the claim payments* is just the present value of the claim payments for each year that death can occur multiplied by the probability that death occurs at that age and then summed for all possible ages. For the term insurance this is summed for ages $x + k$ last birthday where $k = 0, 1, 2, \ldots n - 1$.

Thus the *expected present value of the claim payments for the term insurance is*

$$\begin{aligned} & \sum_{k=0}^{n-1} \frac{S}{(1+i)^{k+1}} \left({}_k p_x q_{x+k} \right) \\ = \; & S \sum_{k=0}^{n-1} v^{k+1} \left({}_k p_x q_{x+k} \right) \end{aligned}$$

The actuarial notation for the *expected present value of the claim payments* for a term insurance on a life aged x with sum insured of \$1 payable on death within n years with the sum insured payable at the end of the year of death is $A^{1}_{x:\overline{n}|}$. The capital A indicates that this is the expected value of a claim (**A***ssurance*) payment. The ^{1} above the age indicates it is paid on the occurrence of death if this occurs first of the two possible events indicated $x : \overline{n}|$ - death at age x or the expiry of n years.

Using the standard actuarial notation we can write

$$A^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} \left({}_k p_x q_{x+k} \right)$$

Example 8.5 *Determine the expected present value of the claim payments for a 5 year term insurance on a life aged 20 with sum insured \$100000 using the following mortality rates and a 6% p.a. effective interest rate.*

Age	q_x
20	*0.00192*
21	*0.00181*
22	*0.00160*
23	*0.00138*
24	*0.00118*

Solution 8.5 *We will assume that death claims are paid at the end of the year of death. In order to determine the expected present value we need to determine* $v^{k+1} = \left(\frac{1}{1.06}\right)^{k+1}$ *and* ${}_kp_x$ *for* $k = 0, 1, 2, 3,$ *and* 4.
The following table shows these values along with the calculation of the expected present value of claim payments for $1.

Age	qx	k	vk+1	kp20	Probability = kp20q20+k
20	0.00192	0	0.9433962	1.0000000	0.0019200
21	0.00181	1	0.8899964	0.9980800	0.0018065
22	0.00160	2	0.8396193	0.9962735	0.0015940
23	0.00138	3	0.7920937	0.9946794	0.0013727
24	0.00118	4	0.7472582	0.9933068	0.0011721
				Expected value	0.0067206

*The expected present value of the claim payments for a sum insured of $100000 is 100000*$\times 0.0067206 = 672.06$.
The values for ${}_kp_x$ *are determined by taking* ${}_0p_x = 1$ *and then calculating* ${}_kp_x$ *using the relationship that*

$$\begin{aligned} {}_{k+1}p_x &= {}_kp_x\left(p_{x+k}\right) \\ &= {}_kp_x\left(1 - q_{x+k}\right) \end{aligned}$$

for $k = 1, 2, 3,$ *and* 4.
Thus

$$\begin{aligned} {}_1p_{20} &= {}_0p_{20} \times (1 - q_{20}) \\ &= 1 \times (1 - 0.00192) \\ &= 0.99808 \end{aligned}$$

$$\begin{aligned} {}_2p_{20} &= {}_1p_{20} \times (1 - q_{21}) \\ &= 0.99808 \times (1 - 0.00181) \\ &= 0.9962735 \end{aligned}$$

The values for v^{k+1} *can also be calculated in a similar recursive approach by noting that*

$$v^{k+1} = \frac{v^k}{1+i}$$

Thus

$$\begin{aligned} v &= \frac{1}{1.06} \\ &= 0.9433962 \end{aligned}$$

$$\begin{aligned} v^2 &= \frac{v}{1.06} \\ &= \frac{0.9433962}{1.06} \\ &= 0.8899964 \end{aligned}$$

and so on.
Thus we can calculate all of the v^{k+1} values by dividing each previous value by $1+i$. The expected value of the claim payments is the sum of the present values at the end of each possible age at death times the probability of payment. This is

$$\begin{aligned} & (0.9433962 \times 0.0019200) \\ & + (0.8899964 \times .0018065) \\ & + (0.8396193 \times 0.0015940) \\ & + (0.7920937 \times 0.0013727) \\ & + (0.7472582 \times 0.0011721) \\ = \; & 0.0067206 \end{aligned}$$

Exercise 8.3 *Determine the expected present value of the claim payments for a 5 year term insurance on a life aged 40 with sum insured \$100000 using the following mortality rates and a 5% p.a. effective interest rate. Also determine the standard deviation of the present value of the claim payments.*

Age	q_x
40	*0.00174*
41	*0.00190*
42	*0.00209*
43	*0.00230*
44	*0.00254*

Premiums The premiums for a term insurance are paid in advance as long as the life is alive, but only for the term of the policy. For simplicity we assume that the premiums are paid annually. This is often the case for term insurance although premiums can be paid monthly or for other payment frequencies during the year.

Denote the premium by P.

If the life is alive at exact age $x+k$ for $k=0,1,2,\ldots n-1$ then a premium will be paid. The present value of the premium due at age $x+k$ is

$$\begin{cases} Pv^k & \text{for } k=0,1,2,\ldots n-1 \\ 0 & \text{for } k \geq n \end{cases}$$

For a life aged x the probability that they will be alive at age $x+k$ and will pay the premium then due is ${}_kp_x$.

The **expected present value of the premiums** at age x is the present value of the premium paid at age $x+k$ multiplied by the probability that the life is alive at age $x+k$ summed for all ages. This is

$$P\sum_{k=0}^{n-1} v^k \left({}_kp_x\right)$$

The actuarial notation for the expected present value of a premium of \$1 for a life aged x paid in advance for a term of n years is

$$\ddot{a}_{x:\overline{n}|}$$

Thus the expected present value of the premiums on the n year term insurance for a life aged x is

$$P\ddot{a}_{x:\overline{n}|}$$

Example 8.6 *Determine the expected present value of premiums of 1 p.a. payable in advance for a 5 year term insurance on a life aged 20 using the following mortality rates and a 6% p.a. effective interest rate (data is as for the previous example).*

Age	**q**$_x$
20	*0.00192*
21	*0.00181*
22	*0.00160*
23	*0.00138*
24	*0.00118*

Solution 8.6 *The following table shows the values used to calculate the expected present value of premiums of 1 p.a. payable in advance*

Age	qx	k	vk	kp20
20	0.00192	0	1.0000000	1.0000000
21	0.00181	1	0.9433962	0.9980800
22	0.00160	2	0.8899964	0.9962735
23	0.00138	3	0.8396193	0.9946794
24	0.00118	4	0.7920937	0.9933068
			Expected value	4.4502088

The expected present value is the sum of the product of the present value of each premium payment of \$1 times the probability that it will be paid.
The present value of the premium paid at the start of the policy is 1 since it is paid at the start of the first year (time 0). The probability that the life is alive when the policy is issued at time 0 is 1.
The present value of the premium payment at the start of the second year is

$$v = \frac{1}{1.06} = 0.9433962$$

and the probability that the life will be alive to pay this premium is

$$p_{20} = 1 - q_{20} = 0.99808$$

We just repeat his for each premium payment. There are five paid at the start of each year of the policy term.
The expected value is just the sum of the present value of the premium times the probability the life will be alive to pay the premium for each premium. This is

$$\begin{aligned} & (1.0000000 \times 1.0000000) \\ & + (0.9433962 \times 0.9980800) \\ & + (0.8899964 \times 0.9962735) \\ & + (0.8396193 \times 0.9946794) \\ & + (0.7920937 \times 0.9933068) \\ = \ & 4.4502088 \end{aligned}$$

Exercise 8.4 *Determine the expected present value of premiums of 1 p.a. payable in advance for a 5 year term insurance on a life aged 40 using the following mortality rates and a 5% p.a. effective interest rate. Also determine the standard deviation of the present value of premium payments of \$1.*

Age	q_x
40	*0.00174*
41	*0.00190*
42	*0.00209*
43	*0.00230*
44	*0.00254*

Recurrence relations In Chapter 4, recurrence relations were used to determine the formula for the expected present value for a life annuity.

The expected present value of benefits for an n year term insurance on a life aged x can also be calculated using recursion. At age $x + n$ the expected value of the benefit will be zero since the benefit is only paid up to age $x + n$.

Let ${}_tB_{x:\overline{n}|}$ be the expected present value of benefits at age $x+t$. This can be related to the expected present value of benefits at $x+t+1$ as follows.

If the life dies during the year, with probability q_{x+t}, then the benefit of S is paid at the end of the year. The expected present value of the benefits will then be equal to S since no future payments are made once the life has died.

If the life survives to the end of the year, with probability p_{x+t}, then the expected present value of the benefits will equal that for a life aged $x+t+1$. This is just ${}_{t+1}B_{x:\overline{n}|}$.

Now these expected values are as at the end of the year. To derive the expected present value at the start of the year using the end of year expected present values we need to divide the end of year value by $1+i$.

We then have

$$ {}_tB_{x:\overline{n}|} = \frac{q_{x+t}S + p_{x+t}\left[{}_{t+1}B_{x:\overline{n}|}\right]}{1+i} $$

The benefit of using the recurrence approach to valuation is that it can easily be programmed in a computer program. All that is needed is the terminal value and then a do loop is used to derive the value at each earlier age. This can also be set up easily in a spreadsheet.

To illustrate the recurrence approach we can derive the expected present value of benefits for the n year term insurance on a life aged x. At $t=n$ we have ${}_nB_{x:\overline{n}|}=0$ since no further benefits are paid after age $x+n$.

Using the recurrence relation we then have over the final year of the policy

$$ \begin{aligned} {}_{n-1}B_{x:\overline{n}|} &= \frac{q_{x+n-1}S + p_{x+n-1}\,0}{1+i} \\ &= \frac{q_{x+n-1}S}{1+i} \end{aligned} $$

Repeating the recursion for another year we then obtain

$$ \begin{aligned} {}_{n-2}B_{x:\overline{n}|} &= \frac{q_{x+n-2}S + p_{x+n-2}\,\frac{q_{x+n-1}S}{1+i}}{1+i} \\ &= q_{x+n-2}\frac{S}{1+i} + p_{x+n-2}\,q_{x+n-1}\frac{S}{(1+i)^2} \end{aligned} $$

Proceeding for another year gives

$$ \begin{aligned} {}_{n-3}B_{x:\overline{n}|} &= \frac{q_{x+n-3}S + p_{x+n-3}\left[q_{x+n-2}\frac{S}{1+i} + p_{x+n-2}\,q_{x+n-1}\frac{S}{(1+i)^2}\right]}{1+i} \\ &= q_{x+n-3}\frac{S}{1+i} + p_{x+n-3}\,q_{x+n-3}\frac{S}{(1+i)^2} + p_{x+n-3}\,p_{x+n-2}\,q_{x+n-1}\frac{S}{(1+i)^3} \\ &= q_{x+n-3}\frac{S}{1+i} + p_{x+n-3}\,q_{x+n-3}\frac{S}{(1+i)^2} + {}_2\,p_{x+n-3}q_{x+n-1}\frac{S}{(1+i)^3} \end{aligned} $$

Continuing backwards in time until age x we have

$$\begin{aligned} {}_0B_{x:\overline{n}|} &= \frac{q_x S + p_x \left[{}_tB_{x:\overline{n}|} \right]}{1+i} \\ &= q_x v S + p_x \, q_{x+1} v^2 S \ldots +_{n-1} p_x q_{x+n-1} v^n S \\ &= S \sum_{k=0}^{n-1} v^k \left({}_k p_x q_{x+k} \right) \end{aligned}$$

The recurrence approach can be used for determining the expected present value of premiums, benefits or other values as well.

Exercise 8.5 *Write a computer program to calculate the expected present value of claims payments for a term insurance with payments at the end of the year of death for input values of S, i, x and n. Read the q_x values from a file or a table. Use the recursive approach and a do loop to calculate the expected present value of claims payments.*

Principle of Equivalence

The premium for a term insurance policy can be determined using the principle of equivalence. An allowance is made for initial expenses and renewal expenses.

Example 8.7 *Use the principle of equivalence to determine the premium for a 5 year term insurance on a 20 year old male with sum insured of $100000 using the following mortality rates and a 6% p.a. effective interest rate (data is as for the previous examples).*

Age	q_x
20	*0.00192*
21	*0.00181*
22	*0.00160*
23	*0.00138*
24	*0.00118*

Allow for initial expenses of 0.5% of the sum insured and renewal expenses of $100 per premium payment.

Solution 8.7 *Denote the premium by P. In the previous examples we have calculated the expected present value of claim payments and premiums at 6% p.a. interest for these mortality rates.*
The principle of equivalence equates the expected present value of premium income to

the expected present value of claims and expenses. Using actuarial notation, for this term insurance policy we have the expected present value of premiums

$$P\ddot{a}_{20:\overline{5}|} = P \times 4.4502088$$

The expected present value of the claims payment will be

$$\begin{aligned} 100000A^{1}_{20:\overline{5}|} &= 100000 \times 0.0067206 \\ &= 672.06 \end{aligned}$$

The expected present value of the initial and renewal expenses will be

$$\begin{aligned} & 0.005 \times 100000 + 100\ddot{a}_{20:\overline{5}|} \\ = \; & 500 + 100 \times 4.4502088 \\ = \; & 945.02 \end{aligned}$$

The premium is then equal to

$$\begin{aligned} P &= \frac{672.06 + 945.02}{4.4502088} \\ &= 363.37 \end{aligned}$$

Exercise 8.6 *Use the principle of equivalence to determine the premium for a 5 year term insurance on a life aged 40 using the following mortality rates and a 5% p.a. effective interest rate.*

Age	q_x
40	*0.00174*
41	*0.00190*
42	*0.00209*
43	*0.00230*
44	*0.00254*

Allow for initial expenses of 0.5% of the sum insured and renewal expenses of $100 per premium payment.

From economic theory we know that in a competitive market the profit will be maximised by charging a premium that will equate marginal cost and marginal revenue. The number of policies sold will depend to some extent on the premium rate charged. Higher premiums will mean less policies are likely to be sold. Since revenue is the product of the number of policies sold times the premium rate per policy there will be an optimum number of policies that will maximise profit.

Some of the expenses of running an insurance company are fixed costs. Premiums in a competitive market will include a loading for marginal costs. Microeconomics studies the theory of the firm and profit maximisation for different types of competition in a market ranging from monopoly to perfect competition.

A loading for profit will also be included. The profit loading will be required to provide shareholders with a competitive return on their equity investment in the company. This loading can be determined using profit testing based on cash flow projection. Allowance for surrenders and lapses should also be made in a profit test. Profit tests are studied in later actuarial subjects.

Probability of ruin

To show how to assess the probability of ruin for an insurance company we will consider a simplified insurance company with n one-year term insurance policies all on lives aged x.

The following assumptions are made

- The probability that each policy will claim during the year is q_x.
- Each policy has sum insured of L.
- All the lives are independent.
- Expenses other than claims are ignored.
- Investment earnings are ignored.

We will only consider the underwriting profit before investment earnings. In practice investment earnings will also contribute to profit.

The probability that there will be j claims in a year from the n policies will have a binomial distribution with

$$\Pr(j \text{ claims}) = \binom{n}{j} q_x^j (1 - q_x)^{n-j} \quad j = 0, 1, 2 \ldots$$

Denote the total claims by T. Let X_i take the value L if the i^{th} life dies during the year or 0 if the life survives. We have

$$T = \sum_{i=1}^{n} X_i = jL$$

The expected value of the total claims in a year will be

$$\begin{aligned} E[T] &= E[jL] \\ &= E[j]L \\ &= nq_x L \end{aligned}$$

The variance of the total value of claims in a year will be

$$\begin{aligned} Var[T] &= Var[jL] \\ &= Var[j]L^2 \\ &= nq_x(1-q_x)L^2 \end{aligned}$$

The second line follows since

$$\begin{aligned} Var[jL] &= E[jL - E[jL]]^2 \\ &= E[jL - E[j]L]^2 \\ &= E[(j - E[j])L]^2 \\ &= E[(j - E[j])]^2 L^2 \\ &= Var[j]L^2 \end{aligned}$$

The company receives total premiums of nP since n policyholders each pay a premium of P. The capital invested by shareholders is assumed to be an amount C. The probability that the company will not be able to pay the claims will be the probability that the total claims exceed the premiums plus capital.

This is

$$\Pr[T > nP + C]$$

If we divide the total claims T by the size of each claim L then $\frac{T}{L}$ is the number of claims j. The probability of ruin is therefore

$$\Pr\left[j > \frac{nP + C}{L}\right]$$

Example 8.8 *An insurance company sells one year term insurance policies on lives aged 20 for a premium of \$250 each with a sum insured of \$100,000. It has sold 10000 independent policies. The probability of a claim on a policy is 0.00192. The company has capital of \$500,000. Determine the probability that the company will not have sufficient funds to pay its total claims.*

Solution 8.8 *Total premium income is*

$$10000 \times 250 = 2,500,000$$

The premiums plus capital available to meet claims is

$$2,500,000 + 500,000 = 3,000,000$$

Since each claim is for $100,000, there is sufficient funds to meet a maximum of 30 claims. The probability that there will not be sufficient funds to meet the claims is the probability that the number of claims exceed 30 or (1-probability that the number of claims is less than or equal to 30).
We know that under in this case the number of claims has a $Binomial(10,000, 0.00192)$ *distribution. We can calculate*

$$1 - \Pr[j \leq 30]$$

using the Binomial distribution in Excel. This is

$$1 - BINOMDIST(30, 10000, 0.00192, TRUE) = 0.00796$$

i.e. less than 1% chance.

Exercise 8.7 *An insurance company sells one year term insurance policies on lives aged 40 for a premium of $220 each with a sum insured of $100,000. The probability of a claim on a policy is 0.00174. The company has sold 10000 policies. Determine the capital that the company requires if the probability that the company will not have sufficient funds to pay its total claims is to be less than 0.001.*

We can approximate the binomial distribution using either a Poisson distribution or a Normal distribution for large values of n. For the normal distribution we can use the mean and variance of the number of claims from the binomial distribution to calculate the required probability. The expected number of claims for the simplified case we are considering is

$$nq_x$$

and the variance of the number of claims is

$$nq_x (1 - q_x)$$

Recall that if a normal random variable X has expected value (mean) μ and standard deviation σ (variance of σ^2) then

$$Z = \frac{X - \mu}{\sigma}$$

has a **standard normal distribution with expected value (mean) of zero and standard deviation of 1.**

Reversing the procedure we can start from a standard normal random variable Z and obtain a normal random variable X with any specified expected value μ and variance σ^2 using

$$X = \mu + \sigma Z$$

The required probability of ruin, assuming that the number of claims has a normal distribution, is

$$\begin{aligned}
& 1 - \Pr\left[j \le \frac{nP + C}{L}\right] \\
= \; & 1 - \Pr\left[\frac{j - nq_x}{\sqrt{nq_x\left(1 - q_x\right)}} \le \frac{\frac{nP+C}{L} - nq_x}{\sqrt{nq_x\left(1 - q_x\right)}}\right] \\
= \; & 1 - \Pr\left[Z \le \frac{\frac{nP+C}{L} - nq_x}{\sqrt{nq_x\left(1 - q_x\right)}}\right]
\end{aligned}$$

where Z is a standard normal random variable with expected value zero and variance equal to 1.

We use the normal distribution with parameters

$$\mu = nq_x$$

and

$$\sigma = \sqrt{nq_x\left(1 - q_x\right)}$$

The probability values for the standard Normal distribution are tabulated in standard statistical tables and can be looked up whenever a computer package with the cumulative distribution for the binomial distribution is not available. The Normal distribution is an approximation.

Example 8.9 *For the previous example, use the normal approximation to estimate the probability that the company will not have sufficient funds to pay its total claims.*

Solution 8.9 *We require*

$$\Pr\left[j > \frac{nP + C}{L}\right]$$

where j *is the number of claims and* $n = 10000$, $P = 250$, $C = 500000$ *and* $L =$

100000.
The probability required is

$$\Pr[j > 30]$$

Now for the binomial distribution we have

$$\begin{aligned}\mu &= nq_x \\ &= 10000 \times 0.00192 \\ &= 19.2\end{aligned}$$

and

$$\begin{aligned}\sigma &= \sqrt{nq_x(1-q_x)} \\ &= \sqrt{10000(0.00192)(1-0.00192)} \\ &= \sqrt{19.163136} \\ &= 4.37757\end{aligned}$$

The required normal approximation is

$$\begin{aligned}& 1 - \Pr\left[Z \leq \frac{30-19.2}{4.37757}\right] \\ = \; & 1 - \Pr\left[Z \leq \frac{30-19.2}{4.37757}\right] \\ = \; & 1 - \Pr[Z \leq 2.46712]\end{aligned}$$

where Z is a standard normal random variable.
From standard normal distribution tables we have

$$\Pr[Z \leq 2.46] = 0.9931$$

and

$$\Pr[Z \leq 2.47] = 0.9932$$

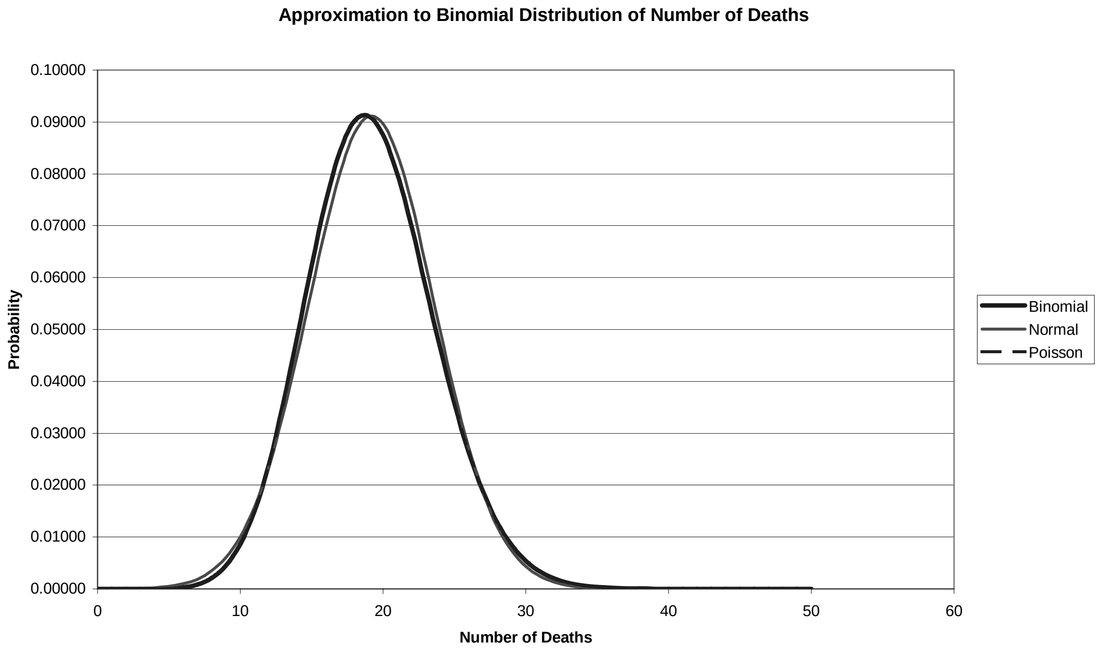
Using ***linear interpolation*** *we obtain*

$$\begin{aligned}\Pr[Z \leq 2.46712] &\doteqdot 0.9931 + \frac{712}{1000}[0.9932 - 0.9931] \\ &= 0.99317\end{aligned}$$

The required probability is

$$1 - 0.99317 = 0.00683$$

The binomial probability was 0.00796. So the normal approximation is close.
The following graph shows the probability density of the binomial, Poisson and normal distributions for the number of claims in this example.



Note that if the claims do exceed the premium income and capital available to pay the claims then the company will be *insolvent* and the policyholders will not receive all of their sum insured. This means that the total claims paid to policyholders who have incurred a loss will be the lesser of the premiums plus capital or the total claims.

If actual claims exceed the funds available of premiums plus capital then the claims paid will be the lesser amount of the premium and capital not the full claims due. This is because the shareholders in a company have *limited liability*. Limited liability means that shareholders of a company are only responsible for the debts of the company to a maximum of their share of the equity of the company.

Worst case for shareholders is that they lose all of their investment but they are not responsible for any further losses of the company not covered by the equity in the company.

Thus claims paid to policyholders will only be

$$\min[T, nP + C]$$

The expected profit for shareholders will be expected premiums less expected claims which is

$$\begin{aligned} & E[nP - \min[T, nP + C]] \\ = \quad & E[nP - T + \max[0, T - (nP + C)]] \end{aligned}$$

Once the total claims T exceed the funds available, $nP + C$, then the policyholders will have to meet the difference, $T - (nP + C)$, out of their own resources.

Simulation

For more realistic assumptions about the claims distributions and for larger, more complex portfolios of policies the determination of probability of ruin, expected profit and other financial variables can best be done using simulation. The use of simulation is covered in more detail in later actuarial subjects.

8.4.2 Valuation of Policy Liabilities

The policy value of a liability for a life insurance policy is the expected present value of future claims and expenses less the expected present value of future premiums as at the date of the liability.

Liability values are determined at the end of each accounting period and are used to determine the profit from the insurance policy. Realistic determination of profit requires the use of realistic values for future interest rates and mortality rates when valuing liabilities.

The interest rates used to value the policy liabilities should also be consistent with the interest rates used to value the assets of the insurance company.

The expected value of the policy liability can be calculated recursively. Denote the expected value of the policy liability at age $x + t$ for the term insurance on a life aged x for a term of n years with sum insured payable at the end of the year of death by ${}_tV^1_{x:\overline{n|}}$.

If the life dies during the year, with probability q_{x+t}, then the benefit of S is paid at the end of the year. The expected value of the policy liability will then be equal to S since this amount is due and payable on death and becomes the policy liability once the life has died.

If the life survives to the end of the year, with probability p_{x+t}, then the expected value of the policy liability will equal that for a life aged $x + t + 1$.

At the start of the year a premium is received and the policy value along with the premium can be invested at interest for a year.

We then have

$$\left[{}_tV^1_{x:\overline{n}|} + P\right](1+i) = q_{x+t}S + p_{x+t}\left[{}_{t+1}V^1_{x:\overline{n}|}\right]$$

so that

$${}_tV^1_{x:\overline{n}|} = \frac{q_{x+t}S + p_{x+t}\left[{}_{t+1}V^1_{x:\overline{n}|}\right]}{(1+i)} - P$$

At the end of the policy term when $t = n$ we have ${}_nV^1_{x:\overline{n}|} = 0$.

This recursive formula can be readily modified to allow for expenses. If E are the expenses paid at the time the premium is paid then

$${}_tV^1_{x:\overline{n}|} = \frac{q_{x+t}S + p_{x+t}\left[{}_{t+1}V^1_{x:\overline{n}|}\right]}{(1+i)} + E - P$$

Example 8.10 *An insurance company sells 5 year term insurances on 20 year old males with a sum insured of $100000 for a premium of $363.37. Assume the following mortality rates, a 6% p.a. effective interest rate, initial expenses of 0.5% of the sum insured and renewal expenses of $100 per premium payment (data is as for the previous examples).*

Age	q_x
20	*0.00192*
21	*0.00181*
22	*0.00160*
23	*0.00138*
24	*0.00118*

- *Determine the expected value of the policy liability for this term insurance as at the end of each year for the five years of the policy term using a recursive formula.*
- *Comment on the values obtained.*

Solution 8.10

- *The expected value of the policy liability for each age is given in the table below*

Age	qx	Expected Value of Policy Liability
20	0.00192	0.00
21	0.00181	-443.68
22	0.00160	-372.80
23	0.00138	-276.43
24	0.00118	-152.05
		0.00

These values are determined as follows.
At age 24 the value is

$$\begin{aligned} & \frac{0.00118 \times 100000 + (1 - 0.00118) \times 0}{1.06} + 100 - 363.37 \\ = \ & -152.05 \end{aligned}$$

At age 23 the value is

$$\begin{aligned} & \frac{0.00138 \times 100000 + (1 - 0.00138) \times (-152.05)}{1.06} + 100 - 363.37 \\ = \ & -276.43 \end{aligned}$$

At age 22 the value is

$$\begin{aligned} & \frac{0.00160 \times 100000 + (1 - 0.00160) \times (-276.43)}{1.06} + 100 - 363.37 \\ = \ & -372.80 \end{aligned}$$

At age 21 the value is

$$\begin{aligned} & \frac{0.00181 \times 100000 + (1 - 0.00181) \times (-372.80)}{1.06} + 100 - 363.37 \\ = \ & -443.68 \end{aligned}$$

Finally at age 20 the value is

$$\begin{aligned} & \frac{0.00192 \times 100000 + (1 - 0.00192) \times (-443.68)}{1.06} + 500 + 100 - 363.37 \\ = \ & 0 \end{aligned}$$

where we have also included the initial expenses for the policy.

- *In this case we see that the expected value of the policy liability is* ***negative****. This means that the liability is effectively an* ***asset*** *for the life insurance company. The reason that it is an asset is twofold. The first reason is that the initial expenses are incurred at the commencement of the policy but are repaid out of the premiums over five years.*
  This is treated as a loan to the policyholder and the amount of unpaid initial expenses is taken into account as an asset in the value of the policy liability.
  Even if the initial expenses are set to zero the policy liability is still negative. In the table below the values are calculated with the initial expenses set to zero.

Age	qx	Expected Value of Policy Liability
20	0.00192	0.00
21	0.00181	-31.98
22	0.00160	-54.92
23	0.00138	-58.23
24	0.00118	-39.70
		0.00

  The values are still negative because the probability of death is ***decreasing*** *over this age range. The premium is level so that in the early years the premium will be too* ***low*** *compared to the expected cost of the death cover. The excess cost of claims in the early years of the policy must be met from shareholders funds or from other policyholder funds. It is recovered later in the policy term when the premium exceeds the cost of death cover. Thus this is like a loan to the policyholders.*
  Insurance legislation does not usually allow life insurance companies to treat a negative liability value as an asset. This is simply because there is no guarantee that the policyholder will pay the later premiums. If the policyholder lapses the policy then they will not pay any negative policy value to the life insurance company. Term insurance contracts do not have surrender values or values on lapse. They certainly do not require the policyholder to pay any unrecouped initial expenses nor any expected death claims costs that have not been charged in the level premium.

Exercise 8.8 *An insurance company sells 5 year term insurances on a 40 year old male with a sum insured of \$100000 for the premium determined using the principle of equivalence. Assume the following mortality rates, a 5% p.a. effective interest rate, initial expenses of 0.5% of the sum insured and renewal expenses of \$100 per premium payment (as for the previous exercises).*

Age	q_x
40	*0.00174*
41	*0.00190*
42	*0.00209*
43	*0.00230*
44	*0.00254*

- *Determine the expected value of the policy liability for this term insurance as at the end of each year for the five years of the policy term using a recursive formula.*
- *Comment on the values obtained.*

8.4.3 Other life insurance benefits

We have only considered a term insurance since this product involves only payment of a death benefit. It has also been assumed that benefits are paid at the end of the year of death. In practice it often takes time to make a death benefit payment since there is the need to provide documentation to the insurance company and arrange for payment.

It is not difficult to allow for benefits to be paid immediately on death. In this case the payment date will be the age at death which is a continuous random variable. All of these results for term insurance can be extended to other types of life insurance policy involving life insurance protection. This is covered in later actuarial subjects.

8.4.4 Investment Policy

The actuary has an important role in establishing the investment policy of an insurance company. The assets of the company consist of the accumulated funds of the policyholders and the capital and retained profits of the shareholders. These are mainly invested in financial assets.

The investment policy of the company determines the proportion of the total funds invested in the different asset classes such as shares, property, fixed interest and cash. The investment policy must also specify the maturity term of any fixed interest investments and the currency of any investments.

The investments must be made taking into account the policy liabilities of the insurance company. This requires a projection of future claims payments, expense payments and premium receipts.

The profit of the company will reflect changes in both the value of the assets and the value of the policy liabilities.

An investment policy adopted by some insurance companies, particularly for annuities and other policies with fixed and guaranteed payments, is a *matching investment strategy.* If the cash flows on the assets from maturing investments and investment income are determined so that they occur at the same time and for the same amount as the expected future claims and expenses less premiums in the policy liability cash flows, then the assets and liabilities are considered to be matched.

In practice investments that match the long term cash flows of an insurance companies liability cash flows are not readily available. In these cases an investment strategy of *immunization* is used. Immunization selects investments so that the change in the value of the assets for a small change of interest rates will equal the change in the value of the policy liabilities for the same small change in interest rates.

The determination of the investment policy to match or immunize policy liabilities is studied in more detail in later actuarial subjects.

8.5 Conclusions

This chapter has introduced the main forms of life insurance policy including the traditional whole of life and endowment policies, term insurance, annuities and the unbundled contracts such as investment account and universal life policies.

The actuarial techniques for determining premium rates, policy liabilities and ruin probabilities for a term insurance were introduced. Issues such as profit loadings and investment strategy were briefly mentioned.

8.6 Solutions to Exercises

Ex 8.1 *You will need to look at the proposal form to see what information is requested. Premium rates are often included with insurance offers made by direct marketing.*

Ex 8.2 *We have*

$$\begin{aligned}
\Pr[K=k] &= \Pr[k \le T(x) < k+1] \\
&= \Pr[x+k \le X < x+k+1 | X > x] \\
&= {}_{k|}q_x \\
&= {}_k p_x q_{x+k} \\
&= \frac{l_{x+k}}{l_x}\frac{d_{x+k}}{l_{x+k}} \\
&= \frac{d_{x+t}}{l_x}
\end{aligned}$$

Ex 8.3 *The expected present value is determined using the formula*

$$S\sum_{k=0}^{n-1} v^{k+1}\left({}_k p_x q_{x+k}\right)$$

with S=100,000, an interest rate of 5% p.a. and the other values shown in the table below. The "Prob" column gives the values for ${}_k p_x q_{x+k}$

Age	qx	k	PV	Prob
40	0.00174	0	95238.10	0.00174
41	0.00190	1	90702.95	0.00190
42	0.00209	2	86383.76	0.00208
43	0.00230	3	82270.25	0.00229
44	0.00254	4	78352.62	0.00252

The expected present value is 903.20.
To calculate the standard deviation we first calculate the variance. To do this we calculate the expected value of the square of the present value and then subtract off the square of the expected value i.e.

$$Var(PV) = E\left[PV^2\right] - E\left[PV\right]^2$$

We obtain the following results

The variance is

Age	qx	k	Prob	PV^2
40	0.00174	0	0.00174	9070294785
41	0.00190	1	0.00190	8227024748
42	0.00209	2	0.00208	7462153966
43	0.00230	3	0.00229	6768393620
44	0.00254	4	0.00252	6139132535

Figure 1

$$\begin{aligned} & 77,872,477 - (903.20)^2 \\ = \; & 77,056,712 \end{aligned}$$

and the standard deviation is

$$\sqrt{77,056,712} = 8,778$$

Note that some care is required in calculating the variance since the random variable takes the values zero after the expiry of the term and it would be necessary to allow for these zero values if the standard formula for variance was used.

Ex 8.4 *The expected present value of premiums of \$1 p.a. payable in advance is determined using the formula*

$$\sum_{k=0}^{n-1} v^k \left({}_kp_x\right)$$

The following table gives the values for the formula

Age	qx	k	PV	kp20
40	0.00174	0	1.000000	1.000000
41	0.00190	1	0.952381	0.998260
42	0.00209	2	0.907029	0.996363
43	0.00230	3	0.863838	0.994281
44	0.00254	4	0.822702	0.991994

The expected present value is 4.52947.

Ex 8.5 *The program needs to input values of S, i, x and n from a file or the keyboard A filename for the mortlaity rates should be input and the q_x values read from this file or table. The recursive approach starts with the expected value equal to zero for the oldest age and then steps backwards one age at a time to the present date. A do loop calculates the expected present value of claims payments by discounting by one period and multiplying by the probability of survival.*

Ex 8.6 *The principle of equivalence equates the expected present value of premiums to the expected present value of claims and expenses. We have*

$$P \times 4.52947 = 903.20 + 500 + 100 \times 4.52947$$

So that

$$\begin{aligned} P &= \frac{903.20 + 500}{4.52947} + 100 \\ &= 409.79 \end{aligned}$$

Ex 8.7 *The premium income is*

$$10000 \times 220 = 2,200,000$$

The required probability is the probability that total claims T exceeds the premium income plus capital.
We require

$$\Pr\left[T > 2,200,000 + C\right] < 0.001$$

If we divide by the sum insured of each policy we require the probability that

$$\Pr\left[j > \frac{2,200,000 + C}{100,000}\right] < 0.001$$

where j is the number of claims which has a Binomial(10000, 0.00174) distribution. We need to determine the number of claims k that will make the probability 1-BINOMDIST(k,10000,0.00174,TRUE) less than 0.001. Using Excel we quickly find that 1-BINOMDIST(31,10000,0.00174,TRUE)=0.00107 and 1-BINOMDIST(32,10000,0.00174,TRUE)=0.000546. Thus k=32 and the amount of capital required is given by

$$\frac{2,200,000 + C}{100,000} = 32$$

or C=1,000,000.

Ex 8.8 *Using the recurrence formula we get the following values for the expected value of the policy liability.*

Age	qx	Expected value of policy liability
40	0.00174	0.00
41	0.00190	-374.37
42	0.00209	-258.30
43	0.00230	-155.25
44	0.00254	-67.89
		0.00

Note that we have

$$\begin{aligned} & \frac{0.00254 \times 100000 + (1 - 0.00254) \times 0}{1.05} + 100 - 409.79 \\ = & -67.89 \end{aligned}$$

$$\begin{aligned} & \frac{0.00230 \times 100000 + (1 - 0.00230) \times -67.89}{1.05} + 100 - 409.79 \\ = & -155.25 \end{aligned}$$

$$\begin{aligned} & \frac{0.00209 \times 100000 + (1 - 0.00209) \times -155.25}{1.05} + 100 - 409.79 \\ = & -258.30 \end{aligned}$$

$$\begin{aligned} & \frac{0.00190 \times 100000 + (1 - 0.00190) \times -258.30}{1.05} + 100 - 409.79 \\ = & -374.37 \end{aligned}$$

$$\begin{aligned} & \frac{0.00174 \times 100000 + (1 - 0.00174) \times -374.37}{1.05} + 500 + 100 - 409.79 \\ = \;& 0 \end{aligned}$$

Now note that the values are negative.
If we remove the initial expenses then we get a premium of $299.40. The expected value of the policy liability becomes.

Age	qx	Expected value of policy liability
40	0.00174	0.00
41	0.00190	35.44
42	0.00209	56.69
43	0.00230	60.03
44	0.00254	42.50
		0.00

Now the values are positive. This is what we would normally expect since the risk in increasing for a level premium.
Early in the policy term the premium is higher than the death risk. The policy liability measures the excess of the death risk cost over the future premiums and later in the policy the premium is lower than the death risk cost. Thus the policy liability is positive early in the term.
The effect of the initial expenses is that this can be considered like a loan to the policyholder that is repaid from the premium only if the policyholder pays all of the premiums. The loan is for $500 and the value of the policy liability falls by the expected value of the unpaid initial expenses since the repayment of these initial expenses is included as part of the premium. This loan is not repaid if the policyholder ceases to pay the premiums during the term of the policy so we should be careful about treating a negative policy liability as an asset.

Chapter 9
PROPERTY AND CASUALTY INSURANCE

9.1 Learning Objectives

The main objectives of this chapter are:

- to introduce the main forms of non-life insurance products and highlight the differences between life and non-life products

- to illustrate the main techniques of actuarial management of non-life insurance products including setting premiums and valuation of policy liabilities.

9.2 Non-life Insurance Products

Non life insurance contracts date back to the early contracts of "bottomry". These were loans made to ship and cargo owners where it was agreed that the loan would not be repaid if the ship or cargo was lost at sea. In the 1500's it was the practice for a shipowner or merchant to prepare a statement of the risks of a voyage and those willing to share the risks would add their name to the statement indicating the amount of the risk that they would take. This was the origin or the term "underwriter" since they would write their names under the statement with the amount of risk they were prepared to cover.

Non-life insurance products provide insurance for property and liability. In North America non-life insurance products are referred to as *property and casualty* insurance products.

Property insurance covers the loss arising from damage to property such as buildings, contents, motor vehicles, aircraft and cargo.

Liability insurance covers the liability to provide compensation to another party when the insured is at fault or where compensation is required by law. Liability insurance can include liability for damage to property and injury to persons.

Some common forms of insurance policy are domestic buildings and contents, motor insurance, public and products liability, and workers' compensation. These are also called *classes* of insurance business.

There are many other forms of general insurance policy covering a range of different risks. Policies cover loss or damage to an insured property (own damage), liability for damage or loss to the property of others (third party property) and liability to third parties for injury.

In many countries certain types of insurance are required by law. For example, in Australia third party motor vehicle insurance is compulsory as is workers' compensation. These classes of insurance that are required by law are often referred to as *statutory* classes of business. They are usually liability classes of business and often involve the settlement of claims for many years into the future arising from injury to persons.

9.2.1 Building and Contents Insurance

These policies cover loss to buildings and contents arising from fire, storm, earthquake, water damage, malicious damage and burglary. The cover for buildings and contents can usually be purchased separately. The policy wording defines the specific events covered and, as well as the loss from fire, storm, burglary, often include breakage of glass, fusion of motors and a range of other causes. The policies often do not provide coverage for flood damage. These policies also include personal liability insurance of a minimum of $2 million.

9.2.2 Motor Insurance

There are a range of different coverages available for motor vehicles. *Comprehensive motor vehicle* insurance provides cover for loss or damage due to accident to the insured vehicle. It also provides coverage for liability for damage to the property of third parties.

Compulsory third party or *CTP* covers bodily injury caused by motor vehicles. It is required by law in all States and Territories in Australia and covers vehicle owners for the liability for damage to the driver or third parties.

Third party property damage insurance is also available to cover liability for damage to the property of others caused by a motor vehicle.

9.2.3 Public and Products Liability

This insurance covers claims made by third parties for injury or damage to property for which the insured is legally liable excluding cover such as that required for motor vehicles. Products liability insurance covers claims arising from the use of a manufacturer's products.

9.2.4 Worker's Compensation

In Australia, legislation requires employers to cover their employees for accidental injury at work, while travelling to and from work or for certain work related illnesses. There are both statutory benefits, required by law, and common law benefits if the employee can prove negligence by the employer. Some employers are able to *self insure* their workers' compensation risk. Self insurance would be used for a large employer who can pool a large group of similar risks and finance the claims costs from their own resources. Self insurance can also provide an incentive for the employer to control claims costs through risk management practices in the company.

9.2.5 Main Features of Non-Life Insurance Products

The policy terms of non-life insurance policies are shorter than for life insurance contracts. Most non-life insurance policies are purchased for a one year term.

Despite this the payment of claims under non-life insurance contracts can extend over many years into the future. This is particularly the case for liability policies. For example under a product liability insurance policy it can take many years until damage is recognized and then it may require lengthy court proceedings to establish liability. Once liability is established it may still take a number of years for the benefit payment to be made. Even in the case of an accident where personal injury occurs immediately it can take many years for the final costs of the damage to be determined and then there may be court proceedings to establish liability and the amount of the damages.

Non-life insurance classes of business are often classified as either *short-tail* or *long tail.*

Short tail classes of insurance product are those where the time from when the claim is incurred (an accident happens) to when the claim is settled (cash is paid to the insured) is relatively short such as a period of up to 12 months to 2 years. This is often the case for house contents and motor vehicle damage policies.

Long tail classes are those classes of insurance business where the time from when the claim is incurred to when the claim is fully paid extends over many years. This can be as long as 10 to 25 years for the liability products such as product liability or worker's compensation injury claims.

With life insurance products the life can only claim once whereas with non-life insurance policies the insured can claim more than once. The number of claims is referred to as the loss *frequency.*

The amount of the claim for a life insurance policy is usually well specified in the contract and is either a fixed sum insured or a sum insured plus bonuses. In non-life insurance the amount of the claim can be quite variable and it is essential to use probability densities to model the size of the loss. The size of the loss is also called the *loss severity*

Non-life insurance policies are also at risk for large claims arising from the one event such as a cyclone, fire or earthquake. This is referred to as *catastrophe risk.*

The variability of claims for a non-life insurance company can be quite high since the variability of each of the underlying risks insured by a non-life company is usually higher than for a life insurance risk.

There can also be a larger mismatch between the time that premiums are received and the time when claims are paid for a non-life insurance company, particularly where it has a large volume of long-tail liability business. This will mean that the profits from a non-life company will depend on the investment returns from investing the premium until the claims payments are made. Investment returns are variable and this can add variability to the profits of a non-life company.

9.3 Actuarial Techniques

Actuaries are involved in premium rating, valuation of policy liabilities, investment strategy and financial management for non-life insurance companies. The key factors to be modelled in non-life insurance are *occurrence*, *timing*, and *severity*.

9.3.1 Occurrence

The occurrence of claims is determined by the claim frequency. The frequency is the number of claims that occur during a particular time period. For an insurance policy this will be the term of the contract. Thus if the policies are one year contracts then the frequency per year will be required for determining the number of claims. This is often estimated with a Poisson distribution since the Poisson distribution is a theoretical model for rare events and insurance claims are considered to be rare events.

The number of claims is a discrete random variable. The actual distribution of the number of claims is determined by the experience of the individuals claims. In practice data on the number of claims is collected for a particular class of insurance business and a claim frequency distribution is estimated from this data. The basic idea is to use a theoretical model for the claims frequency, such as the Poisson probability distribution, and to then use the actual claims data to test if the theoretical probability distribution provides a good fit to the data. The data is also used to estimate the expected number of claims since this will be a parameter of the probability distribution. The process of fitting probability distributions and estimating parameters is studied in detail in statistics and later actuarial subjects.

The Poisson distribution is most often used for claim numbers. There are other distributions used for claim numbers in insurance. Klugman, Panjer and Willmot (1998) [10] cover these in detail. We have already used the binomial distribution for the claims numbers in life insurance. As well as the Poisson distribution, the negative binomial distribution is also used in non-life insurance.

Denote the number of claims on an insurance policy in a year by N. The probability function for the Poisson distribution with parameter λ is

$$\Pr[N = n] = \frac{e^{-\lambda}\lambda^n}{n!} \qquad n = 0, 1, 2, \ldots$$

The expected number of claims in a year is λ and the variance of the number of claims is also λ. In practice we do not know the value of the parameter λ. It is necessary to *estimate* its value from actual insurance data. The methods used for estimating parameters of statistical distributions is covered in detail in later actuarial and statistical subjects.

One approach is to use insurance claims data to calculate the actual average claims rate and to then use this as an estimate of the theoretical average claims rate for the Poisson distribution. In practice we would also need to use statistical tests to determine if the Poisson distribution was a good fit to the claims numbers data as well. These statistical tests are covered in later actuarial subjects.

Example 9.1 *Assume that the number of claims in a year on a class of motor vehicle insurance has a Poisson distribution. Use the following data for the number of claims in a particular year to estimate the Poisson parameter λ.*

Number of claims	***Number of drivers***
0	*89,235*
1	*2,321*
2	*300*
3	*0*
4	*2*
5	*1*
6 or more	*0*

Solution 9.1 *The total number of claims for this class of policyholders is*

$$\begin{aligned} & 89235 \times 0 \\ & +2321 \times 1 \\ & +300 \times 2 \\ & +0 \times 3 \\ & +2 \times 4 \\ & +1 \times 5 \\ = \; & 2934 \end{aligned}$$

The total number of drivers is 91,859. The average number of claims they have is

$$\frac{2934}{91859} = 0.03194$$

We could then use $\lambda = 0.03194$ as the Poisson parameter to determine the probability of a claim for this class of drivers.

Exercise 9.1 *The following table gives the actual distribution of the number of policyholders making claims in a year (see Hossack et al p37 - data taken from a paper by Johnson and Hey).*

Number of claims	***Number of drivers***
0	*370412*
1	*46545*
2	*3935*
3	*317*
4	*28*
5	*3*
6 or more	*0*

Determine the average and variance of the number of claims based on this data. To what extent does your answer confirm the Poisson distribution as an appropriate distribution for the claim numbers.

It is a simple matter to simulate the number of claims for an insurance company. Many spreadsheets and numerical computer programs have random number generators that will generate Poisson random numbers. This is the case with Excel, for example. Practically all of these computer programs can generate uniform random numbers. In Excel, if you enter *rand* () in a cell then it will generate a uniform random number in that cell.

Exercise 9.2 *Use the Excel random number generator for the Poisson distribution to generate the number of claims for a group of 91,859 policyholders with* $\lambda = 0.03194$. *Tabulate a histogram of the number of policyholders with simulated claims of 0,1,2,3,4,5 and 6 or more claims.*

The probability function for the *negative binomial* distribution is

$$\Pr[N = n] = \binom{n+r-1}{n} \left(\frac{1}{1+\beta}\right)^r \left(\frac{\beta}{1+\beta}\right)^n \qquad n = 0, 1, 2, \ldots$$

with expected number of claims

$$E[N] = r\beta$$

and variance of the number of claims

$$Var[N] = r\beta(1+\beta)$$

This distribution is also used for insurance claim numbers. It has more flexibility than the Poisson distribution since the variance is different to the mean. This gives an additional parameter to give a better fit to actual claims numbers data.

9.3.2 Timing

The timing of insurance claims is determined by the future date when claims **are paid**. It is important to note that the date that claims occur will differ from the date on which claims are paid. In order for a claim to be paid by the insurance company it must occur during the time that the policy is in force. Thus if a policy is purchased for a year from 1 January to 31 December then the claim must occur during this period for a claim to be paid.

It may take many years for the claim to be paid by the insurance company. It is the date that the claim is paid that must be estimated in determining the cost of claims in present values. This is also a random variable and there are a number of actuarial techniques that have been developed to allow for the future timing of claims payments.

The expected claims costs is the expected present value of future claims. Future claims will need to be present valued from the date of payment to the current date. The present value will need to take into account that the future claim payment date is a random variable. In order to do this it is necessary to know the probability distribution of the time between claims.

For the Poisson distribution, the time between claims has an exponential distribution with parameter $\frac{1}{\lambda}$. If the number of claims N has a $Poisson\,(\lambda)$ distribution then the time between claims will have an $Exponential\left(\frac{1}{\lambda}\right)$ distribution. Thus if T_i is the time between the i^{th} and the $(i+1)^{th}$ claim for $i = 0, 1, 2 \ldots$, then

$$\Pr\left[T_i \leq t\right] = 1 - e^{-\lambda t}$$

Example 9.2 *You have just had a claim on your motor vehicle insurance policy. The probability that you have a motor vehicle claim has a Poisson*(0.5) *probability distribution. Determine the expected time to the next claim and the probability that you will have a claim in the next 6 months.*

Solution 9.2 *The expected time to the next claim is*

$$\frac{1}{0.5} = 2 \text{ years}$$

The probability that you will have a claim in the next 6 months is

$$\begin{aligned} & 1 - e^{-0.5 \times 0.5} \\ = \; & 0.2212 \end{aligned}$$

i.e. more than one chance in five.

Exercise 9.3 *Assume that the time between car accidents for female drivers aged 20 has an exponential distribution with an expected time of 3 years. It has been 3 years since a 20 year old female driver had an accident. What is the probability that she will have a car accident in the next year?*

9.3.3 Severity

The severity of a claim refers to the total amount of the claim in currency. The severity of claims is modelled using loss distributions including the log-normal and Weibull.

There are many different probability distributions available to model insurance claims and more details are covered in Klugman, Panjer and Willmot (1998) [10].

The log-normal distribution is often used for modelling the levels of monetary amounts including asset and liability values. It is often used for the distribution of non-life insurance claim payments.

Recall that the $Log-normal\,(\mu,\sigma)$ distribution is only defined for positive values and has probability density function (pdf):

$$f(x)=\frac{1}{\sigma x\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2}\quad\text{for } 0<x<\infty$$

and

$$E\left[X\right]=e^{\mu+\frac{1}{2}\sigma^2}$$

$$Var\left[X\right]=e^{2\mu+\sigma^2}\left[e^{\sigma^2}-1\right]$$

$$E\left[X^k\right]=e^{k\mu+\frac{1}{2}k^2\sigma^2}$$

Note also that

$$E\left[\ln\left(X\right)\right]=\mu$$

and

$$Var\left[\ln\left(X\right)\right]=\sigma^2$$

Example 9.3 *Assume that total claim payments on a line of insurance business during a year have a* $Log-normal(12.1, 0.06)$ *distribution. Determine the expected value and variance of the total claims. Also determine the probability that the total claims will exceed \$185,000.*

Solution 9.3 *The expected value of the claim payments during a year will be*

$$\begin{aligned} e^{12.1+\frac{1}{2}\times 0.06^2} &= e^{12.1018} \\ &= 180,196 \end{aligned}$$

The variance of the claim payments will be

$$\begin{aligned} e^{2\times 12.1+0.06^2}\left[e^{0.06^2}-1\right] &= e^{24.2036}\left[0.003606\right] \\ &= 117,104,724 \end{aligned}$$

or a standard deviation of 10,821.5.
The probability that the claims payments, X*, will exceed \$185,000 will be*

$$\begin{aligned} \Pr\left[X>185000\right] &= \Pr\left[\ln X>12.128111\right] \\ &= \Pr\left[\frac{\ln X-12.1}{0.06}>\frac{12.128111-12.1}{0.06}\right] \end{aligned}$$

Now, since X *has a log-normal distribution,* $\ln X$ *has a normal distribution with* $\mu = 12.1$ *and* $\sigma = 0.06$*. Thus*

$$\frac{\ln X - 12.1}{0.06}$$

has a standard normal distribution with expected value 0 and variance 1 and these are tabulated in standard normal tables (or can easily be calculated in a spreadsheet such as Excel using the function NORMSDIST(z)).
The required probability is

$$\Pr[Z > 0.468518] = 1 - \Pr[Z \leq 0.468518]$$

From standard normal tables

$$\Pr[Z \leq 0.46] = 0.67724$$

$$\Pr[Z \leq 0.47] = 0.68082$$

Linear interpolation gives a value for $\Pr[Z \leq 0.468518]$ *of*

$$\begin{aligned} & 0.67724 + \frac{0.468518 - 0.46}{0.47 - 0.46} \times (0.68082 - 0.67724) \\ = \; & 0.68029 \end{aligned}$$

The required probability is $1 - 0.68029 = 0.31971$*. The Excel standard normal function gives* $\Pr[Z \leq 0.468518] = 0.680293$.

Exercise 9.4 *Assume that total claim payments on a line of insurance business during a year have a* $Log - normal(8.5, 0.6)$ *distribution. Determine the expected value and variance of the total claims. Also determine the probability that the total claims will exceed $2,900.*

The parameters for the loss distribution must also be fitted to claims data since we do not know the actual probability distribution that generates the loss. We assume that a particular probability distribution will be a reasonable model of actual losses. There are many ways of fitting loss distributions. Loss distributions are studied in detail in later actuarial studies subjects covering insurance risk models.

9.3.4 Premium rating

In order to determine premium rates it is necessary to determine the expected present value of the claims costs allowing for frequency, timing and severity of the future claims. In order to determine the expected present value it is necessary, at least in theory, to use modified probabilities that allow for the risk in the future claims payments. The theory of how to do this correctly requires the application of advanced concepts from probability and financial economics.

It was shown earlier that it is possible to modify probabilities used to calculate expected values in order to include an allowance for risk. It will be assumed that the probabilities used to calculate expected values are the appropriate probabilities that already allow for the risk in the future claims payments.

A simple case of a one year insurance policy where claims occurring during the policy year are settled exactly one year from the date the policy is sold will be considered.

Denote the number of claims by N and the loss distribution for the i^{th} claim by X_i for $i = 1, 2, 3, \ldots N$. The total claim payments is denoted by S. This is also referred to as the *aggregate loss*. The aggregate loss is just the sum of all the claim amounts that occur during the policy year.

We have

$$\begin{aligned} S &= X_1 + X_2 + \ldots + X_N \qquad N = 1, 2, \ldots \\ &= \sum_{i=1}^{i=N} X_i \end{aligned}$$

Note that the number of terms in the sum, N, is a random variable and that if no claims occur with $N = 0$, then $S = 0$. It will be assumed that, given that we know the number of claims takes the value $N = n$, then the losses on each claim have the same probability distribution, such as a log-normal distribution, and the losses are independent. We will also assume that the number of claims is independent of the losses.

The expected claims costs will be the expected value of the aggregate claims

$$E[S] = E\left[\sum_{i=1}^{i=N} X_i\right]$$

In order to evaluate this probability we need to use some conditional probability results. Note that we do not know how many claims will occur during the year. Initially we need to evaluate the expected value of the total claims assuming we know the number of the claims that occur during the year. Then we have to allow for the different number of possible claims that will occur and the probability that these numbers of claims occur.

We can therefore determine the expected aggregate loss as

$$E[S] = \sum_{n=0}^{n=\infty} E\left[\sum_{i=1}^{i=n} X_i | N = n\right] \Pr[N = n]$$

Since the losses are independent of the number of claims and each loss X_i has the same probability density $f_X(x)$, then the conditional probability that the loss is less than or equal to a particular value x given the number of claims is n does not depend on the number of claims. It is

$$\Pr\left[X \leq x | N = n\right] = \Pr\left[X \leq x\right]$$

Therefore

$$\begin{aligned} E\left[\sum_{i=1}^{i=n} X_i | N = n\right] &= \sum_{i=1}^{i=n} E\left[X_i | N = n\right] \\ &= \sum_{i=1}^{i=n} \int_0^{\infty} \left[x_i\right] f_X\left(x\right) dx \\ &= \sum_{i=1}^{i=n} E\left[X_i\right] \\ &= nE\left[X\right] \end{aligned}$$

The last line follows since all the losses have the same expected value $E\left[X\right]$.

We then have

$$\begin{aligned} E\left[S\right] &= \sum_{n=0}^{n=\infty} E\left[\sum_{i=1}^{i=n} X_i | N = n\right] \Pr\left[N = n\right] \\ &= \sum_{n=0}^{n=\infty} nE\left[X\right] \Pr\left[N = n\right] \\ &= E\left[X\right] \sum_{n=0}^{n=\infty} n \Pr\left[N = n\right] \\ &= E\left[X\right] E\left[n\right] \end{aligned}$$

The premium will be the present value of the expected claims cost plus expected expenses since only a single premium is paid for the policy at the start of the policy year (in advance). Expenses should be added to the expected claims costs allowing for marginal costs in a competitive insurance market in order to maximize profits.

If expected present value of expenses to be loaded in the premium are C then the premium paid at the start of the year, using the principle of equivalence would be,

$$\begin{aligned} P &= \frac{E\left[S\right]}{(1+i)} + C \\ &= \frac{E\left[X\right] E\left[n\right]}{(1+i)} + C \end{aligned}$$

where the interest rate used to present value the expected losses needs to be a rate appropriate to present value expected insurance claims.

Example 9.4 *Assume that the number of claims on a class of insurance business has a* $Poisson(0.1)$ *distribution and that the size of the claims have a* $Log{-}normal(7.5, 0.06)$ *distribution. Claims are assumed to be paid at the end of the year. Assume that the number of claims and the size of claims are independent. Initial expenses amount to \$100 and the interest rate used to present value expected aggregate losses paid at the end of the year is 7% p.a.. Determine the premium including expenses for this risk.*

Solution 9.4 *We have*

$$E[n] = 0.1$$

$$\begin{aligned} E[X] &= e^{7.5+\frac{1}{2}0.06^2} \\ &= 1811.3 \end{aligned}$$

The premium including expenses will be

$$\begin{aligned} & \frac{0.1 \times 1811.3}{1.07} + 100 \\ = & \ \$269.3 \end{aligned}$$

Exercise 9.5 *Assume that the number of claims on a class of insurance business has a* $Poisson(0.01)$ *distribution and that the size of the claims have a* $Log{-}normal(10.5, 0.06)$ *distribution. Assume that the claims are paid in 12 months time. Assume that the number of claims and the size of claims are independent. Initial expenses amount to \$200 and the interest rate used to present value expected aggregate losses paid at the end of the year is 10% p.a. effective. Determine the premium including expenses for this risk.*

If the number of claims is assumed to follow a Poisson distribution then the aggregate loss is said to have a *compound Poisson distribution*. Although we can evaluate the expected value of the claims payments to determine the premium it is often necessary to know the whole probability distribution for the aggregate loss. This is required when evaluating reinsurance premiums or premiums for an insurance policy with a deductible (excess).

The probability distribution can be determined for the whole range of loss values using sophisticated numerical techniques. It is also possible to approximate the aggregate loss distribution with a suitable distribution such as the log-normal distribution. If an approximation to the distribution is used then it is important to examine the fit of the approximate distribution to actual claims data before using the distribution in practice. Methods for doing this are studied in later actuarial subjects.

9.3.5 Valuation of Policy Liabilities

In order to determine the profit for the insurance company at the end of its accounting year it is necessary to value the policy liabilities. The value of policy liabilities for a non-life insurance company must value any expected future claims less any unpaid premiums. Since the premiums for non-life insurance policies are paid in advance there will be no unpaid premiums to include in the value of policy liabilities. In fact since the policyholder pays premiums in the current accounting period for coverage of claims in a future accounting period it is necessary for the insurance company to make a provision for the premiums it has received that cover a period in the next accounting period.

For most classes of insurance business claim payments are made after the year that the claim occurs. Non-life insurance companies value their expected future claims for claims that have occurred and make provision for premiums that cover a later accounting period, which is effectively for future claims that have not yet occurred. The value of the expected future claims is called the *outstanding claims provision*. The amount of the premiums that cover a future accounting period is called the *unearned premium provision*. These two provisions make up the *technical reserves* of the non-life insurance company. These technical reserves appear on the balance sheet of the company.

Unearned Premiums

The insurance policies sold by the insurance company will cover the payment of claims that occur from the date the policies are sold to the date of expiry of the policy which is usually a period of a full year. Policies will be sold throughout the year. The accounting period of the insurance company will not coincide with the policy year for these policies. The company can only take into its profit calculations the part of the premium paid by a policyholder that falls in the accounting period.

Assume that the insurance company uses a calendar year from 1 January to 31 December as its accounting period. Any policy sold on 1 January will have the whole premium taken into profit in the current accounting period. A policy sold on 31 December will have none of the premium paid take into profit in the current accounting period (or at most one day's worth of premium).

The amount of the premium covering the policy year that occurs after the end of the accounting period is called the *unearned premium*. A provision for unearned premium is made in the balance sheet of the insurance company at the end of the accounting period. If the exact date of issue of each policy is known then the proportion of the premium that relates to the period after the end of the accounting period can be calculated and included in the unearned premium provision. Sometimes the information about premiums received is only available on a monthly or quarterly basis. In this case the proportion of the premium to be treated as unearned will be based on the number of months or quarters of the policy year that occurs after the accounting period. In this case the assumption is usually made that all the premiums are received half way through the month or quarter.

The premiums taken into account in order to determine the insurance company's profit are the premiums received minus the change in the unearned premium provision. By taking the unearned premium provision at the start of the accounting period and subtracting the unearned premium provision at the end of the accounting period we determine the component of the unearned premium reserve at the start of the accounting period that is earned during the current accounting period. The premiums earned during the accounting period will be

$$\begin{aligned}\text{Earned premium} \quad &= \text{Premiums Received During the Accounting Period} \\ &\quad +\text{Unearned Premium Provision} \\ &\quad \text{at the Start of the Accounting Period} \\ &\quad -\text{Unearned Premium Provision} \\ &\quad \text{at the End of the Accounting Period}\end{aligned}$$

Example 9.5 *The following table shows the annual premiums received by month for a class of insurance business during the year.*

Month	***Premiums received***
January	*112,234*
February	*60,345*
March	*54,780*
April	*115,200*
May	*80,900*
June	*150,755*
July	*16,340*
August	*50,234*
September	*112,600*
October	*90,765*
November	*112,400*
December	*212,000*

The company has a 31 December financial year for its accounts. Determine the unearned premium provision at the end of the financial year. Assume that premiums were paid on average at the mid point of the month.

Solution 9.5 *The following table gives the amounts of unearned premium for each month as at 31 December. Note that for the premiums paid during January it is assumed that they are received in the middle of the month so that 23/24ths of the premium covers the current financial year and 1/24th covers the risk for the first half of January in the following financial year.*

Similarly for each subsequent month.
As another example, the premiums received during July will be assumed to be received half way through July and so they will cover the risk for 11/24ths of a year in the current financial year and 13/24ths of the following financial year (until mid July).

Month	Premiums	Unearned Fraction as 24ths	Unearned Premium
January	$112,234	1	$4,676
February	$60,345	3	$7,543
March	$54,780	5	$11,413
April	$115,200	7	$33,600
May	$80,900	9	$30,338
June	$150,755	11	$69,096
July	$16,340	13	$8,851
August	$50,234	15	$31,396
September	$112,600	17	$79,758
October	$90,765	19	$71,856
November	$112,400	21	$98,350
December	$212,000	23	$203,167
		Total	$650,043

The total of the unearned premium will be $650,043.

Exercise 9.6 *The following table shows the premiums received during each quarter during a financial year. Determine the unearned premium reserve at the end of the year assuming that premiums are paid on average in the middle of the quarter.*

Quarter ending	***Premiums received***
31 March	*512,658*
30 June	*876,900*
30 September	*456,980*
31 December	*510,456*

Outstanding Claims

Claims occurring in the year that the policy covers will usually be settled in future years. These future claims are referred to as *outstanding claims.* Each year of new business is referred to as a *year of origin.* Policies with claims incurred during a particular year of origin will have these claims settled in future accounting years. The year that the claims are paid for those occurring in a given year or origin is called the *development year.*

The actual claims payments are separated by year of origin and for each year of origin by development year. This is called the *run-off triangle* of claim payments. This information is used to forecast outstanding claims.

Denote the claim amounts paid in development year j arising from losses occurring in year of origin i by X_{ij}. The following table shows a run-off triangle assuming that a company started writing insurance policies in year 0. The claim amounts paid in year 0 arising from losses incurred in year 0 is given by X_{00}. In the second year the claims amounts paid arising from the losses incurred in the year 0 will be X_{01} since these claims will be paid in development year 1 of year of origin 0. The claims amounts paid in this same year from the losses incurred in year 1 will be X_{10} since they occur in development year 0 of the year of origin 1. The total claim payments in the second accounting year will be $X_{01} + X_{10}$.

The current accounting year is year 4. The total claims paid in year 4 are the claims in development year 0 from the losses incurred in the current year (year of origin 4), plus the claims in development year 1 from the losses incurred in the previous year (year of origin 3), plus the claims in development year 2 from the losses incurred in year of origin 2, plus the claims in development year 3 from the losses incurred in year of origin 1, plus the claims in development year 4 from the losses incurred in year of origin 0. Thus the total claims paid in accounting year 4 will be

$$X_{04} + X_{13} + X_{22} + X_{31} + X_{40}$$

which is the sum along the diagonal of the run-off triangle.

Development Year/ Year of origin	**0**	**1**	**2**	**3**	**4**
0	X_{00}	X_{01}	X_{02}	X_{03}	X_{04}
1	X_{10}	X_{11}	X_{12}	X_{13}	
2	X_{20}	X_{21}	X_{22}		
3	X_{30}	X_{31}			
4	X_{40}				

The losses that are incurred in any year will depend on the risk exposure in that year. This might be measured by the number of policies sold in that year. In order to adjust for this effect the total claims in any row would be divided by the number of policies sold in the year of origin. Assume that the X_{ij} values in the run-off triangle have been adjusted for the exposure in each year of origin.

Example 9.6 *The following run-off triangle is a modified version of the data given in Hossack et al [9] on page 208.*

	0	1	2	3	4
1995	580222	466679	240001	133601	54912
1996	494534	499293	192227	159007	
1997	551136	509075	238245		
1998	648031	664188			
1999	746003				

Give an explanation of what the amount of \$54,912 represents.

Solution 9.6 *The claims payments in respect of claims that occurred in accounting year 1995 and eventually settled in 1999 were \$54,912.*

The outstanding claims at the end of an accounting period are the claims that will be paid in development years occurring after the current accounting period. At the end of year of origin 4 the outstanding claim payments are given by the claims payments still to be made for each year of origin. This is shown in the following run-off table which shows the outstanding claims as well as the claims paid as at the end of the current accounting period (year of origin 4).

Development Year/ Year of origin	**0**	**1**	**2**	**3**	**4**	**Total Outstanding (not discounted)**
0	X_{00}	X_{01}	X_{02}	X_{03}	X_{04}	0
1	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{14}
2	X_{20}	X_{21}	X_{22}	X_{23}	X_{24}	$X_{23} + X_{24}$
3	X_{30}	X_{31}	X_{32}	X_{33}	X_{34}	$X_{32} + X_{33} + X_{34}$
4	X_{40}	X_{41}	X_{42}	X_{43}	X_{44}	$X_{41} + X_{42} + X_{43} + X_{44}$

Since we have assumed that the claims for each year of origin are all paid by the end of development year 4, there will be no claims outstanding from year of origin 0. There will be only one year more of claim payments arising from year of origin 1 and so on until year of origin 4 which will have 4 more years of claims payments to be made. Each of these future claim payments are random variables and their expected values will be the expected value of future claims. These need to be determined and then present valued to the current accounting period in order to determine the present value of future claims.

The claim payments arising from each year of origin might be expected to have a similar pattern of payment in future development years. Claim payments will also be expected to increase with inflation but this will increase the payments for each accounting year which means that the effect of inflation will appear along the diagonals of the run-off triangle.

There are many ways that actuaries use to estimate the outstanding claims for non-life insurance companies. These include the *chain ladder method*, the *separation method* and the payments per claim incurred method. These methods are studied in later actuarial professional subjects.

We will consider a simple statistical method that can be used for estimating the outstanding claims that is similar to the *separation method*.

We will make the following assumptions:

- claim payments for each year of origin and development year have a log-normal distribution

- claim payments for each year of origin and development year are independent

- the expected value of the logarithm of the claim payments in year of origin 0 and development year 0 is α

- the expected change in the logarithm of the claim payments from one accounting year to the next is given by f_i for each accounting year i.

- for each year of origin, the expected change in the logarithm of the claims payment from development year $j-1$ $(j = 1, 2 \ldots)$ to development j is equal to β_j and this is the same for each year of origin.

- the logarithm of claim payments have the same variance, σ^2, regardless of year of origin or year of development.

The f_i values allow for any inflation in values from one accounting year to the next. The β_j values allow for the settlement pattern of claims over time arising from the same policy year.

The run-off triangle of expected values for the logarithm of the claims payments will then be

Development Year/ Year of origin	**0**	**1**	**2**	**3**	**4**
0	α	$\alpha + f_1 + \beta_1$	$\alpha + f_1 + f_2 + \beta_1 + \beta_2$	$\alpha + f_1 + f_2 + f_3 + \beta_1 + \beta_2 + \beta_3$	$\alpha + f_1 + f_2 + f_3 + f_4 + \beta_1 + \beta_2 + \beta_3 + \beta_4$
1	$\alpha + f_1$	$\alpha + f_1 + f_2 + \beta_1$	$\alpha + f_1 + f_2 + f_3 + \beta_1 + \beta_2$	$\alpha + f_1 + f_2 + f_3 + f_4 + \beta_1 + \beta_2 + \beta_3$	
2	$\alpha + f_1 + f_2$	$\alpha + f_1 + f_2 + f_3 + \beta_1$	$\alpha + f_1 + f_2 + f_3 + f_4 + \beta_1 + \beta_2$		
3	$\alpha + f_1 + f_2 + f_3$	$\alpha + f_1 + f_2 + f_3 + f_4 + \beta_1$			
4	$\alpha + f_1 + f_2 + f_3 + f_4$				

Under these assumptions, we then have that

$$\ln(X_{ij}) \backsim Normal\left(\alpha + \sum_{k=1}^{i+j} f_k + \sum_{k=1}^{j} \beta_k, \sigma^2\right)$$

Given these assumptions we know that

$$E[X_{ij}] = \exp\left[\alpha + \sum_{k=1}^{i+j} f_k + \sum_{k=1}^{j} \beta_k + \frac{1}{2}\sigma^2\right]$$

For the run-off triangle given earlier, if we estimate the parameters f_i for $i = 0, 1, 2, 3, 4$, β_k for $k = 1, 2, 3, 4$ and σ^2 then it will be possible to determine the parameters of the distribution assumed for outstanding claims for each year of origin. Note that we are estimating 10 parameter values using 15 data points.

One method of fitting these parameters is to use least squares. Another method is to use a technique often used in statistics called *maximum likelihood*. Maximum likelihood determines the likelihood of each actual claim value using the probability distribution assumed for the claims. In this case, since we are assuming a log-normal distribution, the likelihood of each X_{ij} is given by

$$\frac{1}{\sigma X_{ij}\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln X_{ij} - \left[\alpha + \sum_{k=1}^{i+j} f_k + \sum_{k=1}^{j} \beta_k\right]}{\sigma}\right)^2\right]$$

The likelihood of all the values is just the product of the likelihoods of each value so the likelihood of the actual run-off triangle values will be

$$\prod_{i=0}^{4}\prod_{j=0}^{4-i} \frac{1}{\sigma X_{ij}\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln X_{ij} - \left[\alpha + \sum_{k=1}^{i+j} f_k + \sum_{k=1}^{j} \beta_k\right]}{\sigma}\right)^2\right]$$

The values for f_i for $i = 0, 1, 2, 3, 4$, β_k for $k = 1, 2, 3, 4$ and σ^2 are then chosen to maximize the likelihood of the values in the run-off triangle. We can maximize the logarithm of the likelihood and obtain the same values since logarithmic transform is monotonic. The logarithm of the likelihood is called the *log-likelihood*.

The log-likelihood for the run-off triangle is

$$\sum_{i=0}^{4}\sum_{j=0}^{4-i} -\ln\left[\sigma X_{ij}\sqrt{2\pi}\right] - \frac{1}{2}\left(\frac{\ln X_{ij} - \left[\alpha + \sum_{k=1}^{i+j} f_k + \sum_{k=1}^{j} \beta_k\right]}{\sigma}\right)^2$$

To maximize this function we select the parameters that minimise

$$\sum_{i=0}^{4}\sum_{j=0}^{4-i} \ln\left[\sigma X_{ij}\sqrt{2\pi}\right] + \frac{1}{2}\left(\frac{\ln X_{ij} - \left[\alpha + \sum_{k=1}^{i+j} f_k + \sum_{k=1}^{j} \beta_k\right]}{\sigma}\right)^2$$

This is similar to what we would have done if we were to use least squares except that we have allowed for the variance parameter to be included in the estimation.

Once the parameters have been estimated it is possible to then estimate the expected future payments allowing for the estimated development of claims. However it will be necessary to estimate future increases by accounting year (i.e. claims payment inflation). This can be estimated using a variety of methods based on the estimated rates of change in the logarithm of the claims payments from the run-off triangle and on current estimates of inflation in claims payments.

Example 9.7 *The following table gives a run-off triangle as at 31 December 1999 for a class of non-life insurance business showing estimates of the expected value of outstanding claims arising from business written in each of the years 1996 to 1999 inclusive. All claims are settled by the end of development year 4.*

	0	1	2	3	4
1995	580222	466679	240001	133601	54912
1996	494534	499293	192227	159007	67185
1997	551136	509075	238245	196972	82060
1998	648031	664188	309630	240582	100229
1999	746003	778777	378183	293848	122420

The following interest rates are to be used to present value the expected value of outstanding claims in order to determine the outstanding claims provision as at 31 December 1999.

Time to payment - years	*Effective Interest Rate % p.a.*
0.5	*5.0*
1.5	*5.5*
2.5	*5.9*
3.5	*6.5*

These interest rates are called a term structure of interest rates and reflect the fact that a higher interest rate normally applies to longer times to receipt of cash flows. Determine the discounted outstanding claims provision assuming claim payments are made on average in the middle of the year of payment i.e. 30 June.

Solution 9.7 *The present value of the expected outstanding claim payments for year of origin 1996 is*

$$\frac{67,185}{(1.05)^{0.5}} = 65,566$$

since we use the 0.5 year interest rate and discount for half a year.
For year of origin 1997 the present value of the expected outstanding claim payments is

$$\frac{196,972}{(1.05)^{0.5}} + \frac{82,060}{(1.055)^{1.5}} = 267,952$$

For year of origin 1998 the present value of the expected outstanding claim payments is

$$\frac{309,630}{(1.05)^{0.5}} + \frac{240,582}{(1.055)^{1.5}} + \frac{100,229}{(1.059)^{2.5}} = 611,031$$

For the final year of origin, 1999, the present value of the expected outstanding claim payments is

$$\frac{778,777}{(1.05)^{0.5}} + \frac{378,183}{(1.055)^{1.5}} + \frac{293,848}{(1.059)^{2.5}} + \frac{122,420}{(1.059)^{3.5}} = 1,461,824$$

This gives a total discounted expected outstanding claims value of 2,406,375.
The following table shows each of the discounted outstanding claims values.

Present values of Expected Outstanding Claims

	0	1	2	3	4	Total
1995						
1996					65566	65566
1997				192225	75728	267953
1998			302168	222016	86847	611031
1999		760008	348999	254614	98204	1461824
						2406375

The claim payments to be brought to profit in the current accounting period must relate to the claims that occur during the accounting period even if the payment of the claims are made in future accounting periods. The claims that relate to a particular accounting period are called the *claims incurred* and are determined by

$$\begin{aligned}\text{Claims incurred} \quad = \quad & \text{Claims Paid During the Accounting Period} \\ & +\text{Outstanding Claims Provision} \\ & \text{at the End of the Accounting Period} \\ & -\text{Outstanding Claims Provision} \\ & \text{at the Start of the Accounting Period}\end{aligned}$$

Profit and Loss

Profit is determined as

$$
\begin{aligned}
& \text{Premiums } + \text{ Investment Income} \\
& -\text{Claims} - \text{Expenses} - \text{Increase in Value of Policy Liabilities} \\
= \; & \text{Earned Premiums} + \text{Investment Income} - \text{Incurred Claims} - \text{Expenses}
\end{aligned}
$$

The increase in the value of the policy liabilities is the increase in the outstanding claims provision plus the increase in the unearned premium provision. Thus

$$
\begin{aligned}
& \text{Increase in value of policy liabilities} \\
= \; & \text{Outstanding Claim Provision} \\
& \text{at the End of the Accounting Period} \\
& -\text{Outstanding Claim Provision} \\
& \text{at the Start of the Accounting Period} \\
& +\text{Unearned Premium Provision} \\
& \text{at End of the Accounting Period} \\
& -\text{Unearned Premium Provision} \\
& \text{at Start of the Accounting Period}
\end{aligned}
$$

Example 9.8 *An insurance company has the following provisions at the end of the financial years shown.*

Provision	***31 December XX-1***	***31 December XX***
Unearned Premium	*1,234,900*	*1,456,200*
Outstanding Claims	*2,500,560*	*2,300,100*

During the year premiums received amounted to 2,000,250, investment income was 230,000, claims paid were 1,500,250 and expenses were 812,000. Determine the profit for the company for the year XX.

Solution 9.8 *The incurred claims were*

$$1,500,250 + 2,300,100 - 2,500,560 = 1,299,790$$

The earned premiums were

$$2,000,250 + 1,234,900 - 1,456,200 = 1,778,950$$

The profit was

$$1,778,950 + 230,000 - 1,299,790 - 812,000 = -102,840$$

which is a loss.

In assessing the loss and expense experience of a non-life insurance company it is common practice to calculate accounting ratios based on the claims incurred and the earned premium. The *loss ratio* is the ratio of incurred claims to earned premiums. It indicates the proportion of the premiums that are paid in meeting claims. Note that the loss ratio does not indicate the loss that the company made. It indicates the proportion of the premium required to meet claims losses.

Example 9.9 *Calculate the loss ratio for the previous example.*

Solution 9.9 *The loss ratio is incurred claims divided by earned premiums:*

$$\frac{1,299,790}{1,778,950} = 0.73$$

Exercise 9.7 *You have extracted the following information from the accounts of a non-life insurance company.*

Item	***Amount***
Unearned Premium Provision 31 December XX-1	*50,007,450*
Unearned Premium Provision 31 December XX	*60,005,500*
Outstanding Claims Provision 31 December XX-1	*90,000,400*
Outstanding Claims Provision 31 December XX	*140,500,800*
Premiums received in year XX	*120,007,000*
Claims paid in year XX	*50,600,900*
Expenses in year XX	*25,000,500*
Investment Income in year XX	*35,120,000*

Determine the profit for the company for financial year XX and the loss ratio.

9.3.6 Reinsurance and Deductibles

Many insurance policies have deductibles or excesses where the policyholder is required to pay the first part of any loss. Thus if a policy has an excess or deductible of D then this means that , for an aggregate loss of S, the insured will have to pay

$$\begin{cases} S & \text{for } S \leq D \\ D & \text{for } S > D \end{cases}$$

for any loss.

The insurance company will pay a claim amount

$$\begin{cases} 0 & \text{for } S \leq D \\ S - D & \text{for } S > D \end{cases}$$

In order to determine premiums for these policies it is necessary to know the probability distribution for the aggregate loss, $F_S(s)$. The probability density will be denoted $f_S(s)$.

The expected claim amount to be paid by the insurance company is

$$\begin{aligned} & \int_D^\infty (s-D)\, f_S(s)\, ds \\ = & \int_D^\infty s f_S(s)\, ds - \int_D^\infty D f_S(s)\, ds \\ = & \int_D^\infty s f_S(s)\, ds - D\int_D^\infty f_S(s)\, ds \\ = & \int_D^\infty s f_S(s)\, ds - D\left[1 - F_S(D)\right] \end{aligned}$$

Example 9.10 *Assume that the aggregate loss has a Log−normal (μ, σ) distribution. Show that the expected claim amount for an insurance policy with an excess of D is*

$$e^{\mu+\frac{1}{2}\sigma^2} N\left(\frac{-\ln D + \mu}{\sigma} - \sigma\right) - DN\left(\frac{-\ln D + \mu}{\sigma}\right)$$

where $N(x)$ is the standard normal cumulative density given by

$$N(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

Solution 9.10 *The expected claim amount is*

$$\begin{aligned} & \int_D^\infty (s-D) \frac{1}{\sigma s\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln s - \mu}{\sigma}\right)^2\right] ds \\ = & \int_D^\infty s \frac{1}{\sigma s\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln s - \mu}{\sigma}\right)^2\right] ds - D\int_D^\infty \frac{1}{\sigma s\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln s - \mu}{\sigma}\right)^2\right] ds \end{aligned}$$

Now change variables to

$$y = \ln s$$

with

$$dy = \frac{1}{s} ds$$

The expected claim amount for this change of variable becomes

$$\int_{\ln D}^\infty e^y \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right] dy - D\int_{\ln D}^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right] dy$$

Consider the first integral in this expression. We have

$$
\begin{aligned}
& \int_{\ln D}^{\infty} e^{y} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^{2}\right] dy \\
= & \int_{\ln D}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[y-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^{2}\right] dy \\
= & \int_{\ln D}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[y-\frac{1}{2}\left(\frac{y^{2}-2y\mu+\mu^{2}}{\sigma^{2}}\right)\right] dy \\
= & \int_{\ln D}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{-2y\sigma^{2}+y^{2}-2y\mu+\mu^{2}}{\sigma^{2}}\right)\right] dy \\
= & \int_{\ln D}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{-2y\sigma^{2}+y^{2}-2y\mu+\mu^{2}}{\sigma^{2}}\right)\right] dy \\
= & \int_{\ln D}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\left[y-(\mu+\sigma^{2})\right]^{2}+\mu^{2}-(\mu+\sigma^{2})^{2}}{\sigma^{2}}\right)\right] dy \\
= & \int_{\ln D}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\left[y-(\mu+\sigma^{2})\right]^{2}-2\mu\sigma^{2}+\sigma^{4}}{\sigma^{2}}\right)\right] dy \\
= & e^{\mu+\frac{1}{2}\sigma^{2}} \int_{\ln D}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y-(\mu+\sigma^{2})}{\sigma}\right)^{2}\right] dy
\end{aligned}
$$

Now make the change of variable

$$
z = \frac{y-(\mu+\sigma^{2})}{\sigma}
$$

with

$$
dz = \frac{dy}{\sigma}
$$

The integral becomes

$$
\begin{aligned}
& e^{\mu+\frac{1}{2}\sigma^{2}} \int_{\frac{\ln D-(\mu+\sigma^{2})}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z)^{2}} dz \\
= & e^{\mu+\frac{1}{2}\sigma^{2}} \int_{-\infty}^{-\frac{\ln D-\left(\mu+\sigma^{2}\right)}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z)^{2}} dz \\
= & e^{\mu+\frac{1}{2}\sigma^{2}} \int_{-\infty}^{\frac{-\ln D+\mu}{\sigma}-\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z)^{2}} dz \\
= & e^{\mu+\frac{1}{2}\sigma^{2}} N\left(\frac{-\ln D+\mu}{\sigma}-\sigma\right)
\end{aligned}
$$

The second line uses symmetry of the normal density.
Considering the second integral we have, making the change of variable

$$z = \frac{y-\mu}{\sigma}$$

and

$$dz = \frac{dy}{\sigma}$$

so that

$$\begin{aligned}
& D\int_{\ln D}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right] dy \\
= \;& D\int_{\frac{\ln D-\mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
= \;& D\int_{-\infty}^{-\frac{\ln D-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
= \;& DN\left(\frac{-\ln D+\mu}{\sigma}\right)
\end{aligned}$$

Thus the expected claim amount for an insurance policy with an excess of D is

$$e^{\mu+\frac{1}{2}\sigma^2} N\left(\frac{-\ln D+\mu}{\sigma} - \sigma\right) - DN\left(\frac{-\ln D+\mu}{\sigma}\right)$$

Exercise 9.8 *An insurance company issues a policy with a deductible of \$100. The policy has an aggregate loss with a* $Log-normal\,(5, 0.5)$ *distribution. Determine the expected value of the claims paid by the insurance company allowing for the deductible.*

Insurance companies also purchase reinsurance of their risks. One form of reinsurance is the excess-of-loss (XOL) reinsurance policy where the excess over an agreed claim amount is paid by the reinsurance company. Assume that the excess-of-loss policy covers claims that exceed D. Thus for an aggregate loss of S, the reinsurance company will pay

$$\begin{cases} 0 \;\text{ for } S \leq D \\ S-D \;\text{ for } S > D \end{cases}$$

This is exactly the same claim payment faced by an insurance company with an excess or deductible in its policies. The insurance company with an excess-of-loss reinsurance coverage for its losses will pay

$$\begin{cases} S \;\text{ for } S \leq D \\ D \;\text{ for } S > D \end{cases}$$

so that its losses are capped at an amount of D.

9.4 Conclusions

This chapter has outlined the main features of non-life insurance contracts. These are often referred to as property and casualty insurance since they cover the loss to property and injury to individuals. The shorter term nature of the contracts was discussed and the methods of determining the profit of non-life insurance business was briefly outlined. This included a brief coverage of the method of setting premium rates and of determining the value of policy liabilities. These areas are covered in more detail in Hossack, Pollard and Zehnwirth [9] as well as in Klugman, Panjer and Willmot [10].

9.5 Solutions to Exercises

Ex 9.1 *Let X be the number of claims. The average or expected number is $E[X]$ which equals*

$$\begin{aligned}
& \frac{1 \times 46545 + 2 \times 3935 + 3 \times 317 + 4 \times 28 + 5 \times 3}{370412 + 46545 + 3935 + 317 + 28 + 3} \\
= & \frac{55493}{421240} \\
= & 0.13174
\end{aligned}$$

The variance is $E[X^2] - E[X]^2$. We have

$$\begin{aligned}
E\left[X^2\right] &= \frac{1^2 \times 46545 + 2^2 \times 3935 + 3^2 \times 317 + 4^2 \times 28 + 5^2 \times 3}{370412 + 46545 + 3935 + 317 + 28 + 3} \\
&= \frac{65661}{421240} \\
&= 0.15588
\end{aligned}$$

The variance is

$$\begin{aligned}
& 0.15588 - (0.13174)^2 \\
= & 0.13852
\end{aligned}$$

For the Poisson distribution the mean and variance should be equal. In this case they are almost equal so the Poisson distribution may be adequate. There are statistical tests covered in later actuarial subjects that can more formally determine the appropriateness of the Poisson distribution.

Ex 9.2 *You need to use the Tools, Data Analysis, Random Number Generation function for a Poisson random variable and then generate a histogram using the Histogram function.*

Ex 9.3 *The required probability is*

$$\Pr[T \leq 4 | T > 3]$$

where T *has an Exponential*(3) *so that* $\Pr[T \leq t] = 1 - e^{-\frac{1}{3}t}$. *We have*

$$\begin{aligned}
\Pr[T \leq 4 | T > 3] &= \frac{\Pr[3 < T \leq 4]}{\Pr[T > 3]} \\
&= \frac{\Pr[T \leq 4] - \Pr[T \leq 3]}{1 - \Pr[T \leq 3]} \\
&= \frac{1 - e^{-\frac{1}{3}4} - \left(1 - e^{-\frac{1}{3}3}\right)}{1 - \left(1 - e^{-\frac{1}{3}3}\right)} \\
&= \frac{e^{-\frac{1}{3}3} - e^{-\frac{1}{3}4}}{e^{-\frac{1}{3}3}} \\
&= \frac{0.10428}{0.36788} \\
&= 0.28347
\end{aligned}$$

Note that we can also determine this as

$$\begin{aligned}
\Pr[T \leq 1] &= 1 - e^{-\frac{1}{3}} \\
&= 0.28347
\end{aligned}$$

Ex 9.4 *The expected value of total claim payments will be*

$$\begin{aligned}
&\exp\left(8.5 + \frac{1}{2}0.6^2\right) \\
&= e^{8.68} \\
&= 5,884
\end{aligned}$$

The variance of total claims will be

$$\begin{aligned}
&e^{2\times 8.5 + 0.6^2}\left[e^{0.6^2} - 1\right] \\
&= 34622004 \times 0.43333 \\
&= 15002733
\end{aligned}$$

(a standard deviation of 3,873.
The probability that the total claims will exceed \$2,900 is

$$\begin{aligned}
\Pr[X > 2900] &= \Pr[\ln X > 7.972466] \\
&= \Pr\left[\frac{\ln X - 8.5}{0.6} > \frac{7.972466 - 8.5}{0.6}\right] \\
&= \Pr[Z > -0.87922]
\end{aligned}$$

which equals 1-$\Pr[Z \leq -0.87922] = 1 - 0.189640064 = 0.81036.$

Ex 9.5 *The premium will be*

$$\frac{E[X]\,E[n]}{(1+i)} + C$$

where

$$\begin{aligned} E[X] &= \exp\left[10.5 + \frac{1}{2}0.06^2\right] \\ &= 36,381 \end{aligned}$$

$$E[n] = 0.01$$

$$C = 200$$

$$i = 0.1$$

So the premium is

$$\begin{aligned} & \frac{36,381 \times 0.01}{(1.1)} + 200 \\ = \;& 530.74 \end{aligned}$$

Ex 9.6 *The unearned premiums will equal*

Quarter ending	***Premiums received***
31 March	$\frac{1}{8} \times 512,658 = 64,082.25$
30 June	$\frac{3}{8} \times 876,900 = 328,837.5$
30 September	$\frac{5}{8} \times 456,980 = 285,612.5$
31 December	$\frac{7}{8} \times 510,456 = 446,649$

Giving a total unearned premium provision of 1,125,181.

Ex 9.7 *The earned premium is*

$$120,007,000 + 50,007,450 - 60,005,500 = 110,008,950$$

The claims incurred is

$$50,600,900 + 140,500,800 - 90,000,400 = 101,101,300$$

The profit is

$$110,008,950 + 35,120,000 - 101,101,300 - 25,000,500 = 19,027,150$$

The loss ratio is

$$\frac{101,101,300}{110,008,950} = 0.919$$

Ex 9.8 *The expected value of the claims paid by the insurance company allowing for the deductible will be*

$$\int_{100}^{\infty} (s - 100)\, f_S(s)\, ds$$

where $f_S(s)$ *is the probability density of a* $Log - normal\,(5, 0.5)$ *distribution. The expected value is*

$$\begin{aligned}
& e^{\mu+\frac{1}{2}\sigma^2} N\left(\frac{-\ln D + \mu}{\sigma} - \sigma\right) - DN\left(\frac{-\ln D + \mu}{\sigma}\right) \\
= \; & e^{5+\frac{1}{2}0.5^2} N\left(\frac{-\ln 100 + 5}{0.5} - 0.5\right) - 100N\left(\frac{-\ln 100 + 5}{0.5}\right) \\
= \; & 168.17N\,(0.28966) - 100N\,(0.78966) \\
= \; & 168.17 \times 0.613962 - 100 \times 0.785137 \\
= \; & 24.7
\end{aligned}$$

Chapter 10
RETIREMENT, SOCIAL SECURITY AND HEALTH CARE FINANCING

10.1 Learning Objectives

The main objectives of this chapter are:

- to introduce the main forms of retirement, social security and health care benefits, and
- to outline some techniques of actuarial modelling used in these areas.

10.2 Retirement and Social Security

Insurance provides financial security in the event of death, injury or damage to property. There are benefits from the pooling of these risks through insurance. Other major risks faced by individuals include the risk that they live for longer than they have resources to support themselves when they retire, the risk of deterioration in health, as well as the risk of becoming unemployed.

Inability to work due to sickness or disability before retirement will usually mean that an individual will not be able to earn the same level of income as when they had their full health. After retirement, at older ages, there is the need to provide for long term care costs including nursing home care as health deteriorates.

Social security benefits provide income for retirement and to provide income during unemployment as well as a range of other benefits. The taxation system is used to finance the costs of various social security benefits as well as hospital and medical benefits. In some countries the government applies a specific levy on income of individuals to finance the costs of social security and retirement benefits provided by the government. In other countries these benefits are financed from overall taxation revenue. Taxation revenue is also used to finance public hospital and medical costs.

Example 10.1 *How are health costs financed in Australia?*

Solution 10.1 *In Australia, the Medicare system provides coverage for hospital and medical treatment and is financed partly from a direct Medicare levy, but mostly from taxation revenue, since the costs of health care under this system far exceed the Medicare levy collections.*

Improvements in mortality rates have resulted in individuals living longer. As a result they require income to live on after they cease working for increasing lengths of time. The risk that individuals live longer than on average is called *longevity* risk.

Increasing amounts of money will be required to support an ageing population since the demands for retirement and long term care are forecast to increase significantly over the next 40 years. Most developed countries now face an ageing population.

Improved mortality and lower fertility rates mean that the *dependency ratio,* defined as the ratio of those not in the work force (under age 15 and over age 65) to those in the working ages 15 to 64, is increasing. This means that the population of working age will have to pay increased taxes to support an increasing number of dependents.

Example 10.2 *What changes are expected in dependency ratios in OECD countries?*

Solution 10.2 *In the OECD countries the dependency ratios will increase by as much as 40% over the next 40 years from about 0.5 currently to about 0.7 in 2030.*

Retirement Income

There are three levels or *pillars* for retirement provision.

The *first pillar* is the government provided pension, the *second pillar* is the provision made for retirement through employer and other approved retirement schemes and the *third pillar* is the private savings of individuals in addition to their employer and other approved superannuation funds.

In many countries, the government provides an age pension to those over a specified retirement age, usually 65, as the first pillar. In some countries this is financed by a separate levy during the working life of individuals. In other countries this is financed from the revenue of the government at the time of payment and there is no specific levy for the retirement pension.

Example 10.3 *Explain how government provided pensions are determined in Australia.*

Solution 10.3 *In Australia the government provides an aged pension of approximately 25% of average weekly earnings. A single aged pensioner in Australia in 1999 receives a maximum of $366.50 per fortnight and a married couple each receives $305.90 per fortnight. These pension payments are subject to means tests. A means test reduces the pension if an individual has greater means. The more assets and income that an individual has, the less the age pension that they receive. For example in Australia, a single individual with income of more than $102 per fortnight would have their pension reduced by 50c for every dollar of income over this figure. The pension would cease if they had an income of more than $845.80 per fortnight. Similarly, in Australia, if a single pensioner owns a home, then if their assets exceeds*

$127,750 then their pension would reduce until they would receive no pension if they have more than $251,750 in assets. Similar income and assets tests also apply for married couples and for non-home owners.

The second pillar of retirement provision is the superannuation contributions provided by employers for their employees and by the self employed. These contributions are now compulsory in Australia. The government has mandated, under the Superannuation Guarantee, that employers must provide a superannuation contribution of at least 9% of salary for their employees. Although the government requires that these contributions be made by an employer, the benefits are provided by private superannuation funds and not government superannuation funds. The contributions can be made into an *employer sponsored superannuation fund* or into an approved fund such as a fund sponsored by a union.

In some countries, such as Singapore and Hong Kong, the government has established a Central Provident Fund which invests the compulsory employer retirement contributions and is responsible for the payment of future retirement benefits.

Benefits from employer sponsored superannuation funds can be paid as either a *pension* or a *lump sum*.

Pensions are paid from a superannuation fund as an income when a fund member retires as long as they are alive. If the retiree has a partner then the pension usually continues being paid to the partner on the death of the retiree, but at a reduced level.

Lump sums are paid as a single benefit amount from a superannuation fund on the retirement date of the member. No further payments are made from the fund to the member once they retire. The member must then invest the lump sum to provide for their retirement. One method of providing an income in retirement when a lump sum is received is to purchase a life annuity from a life insurance company.

The benefits provided from employer superannuation funds can be either *defined benefit* or *defined contribution* benefits. Defined benefit (DB) funds provide retirement benefits that are specified by a formula and are related to salary prior to retirement. The superannuation fund has a *Trust Deed* which specifies the formulae used to determine the benefits of the fund including the benefit paid on resignation. Benefits payable on death or ill health retirement would also be specified as a multiple of salary in a defined benefit fund. A typical retirement benefit would be a final average salary benefit paid as a pension or a lump sum based on the average salary, or highest average salary, over the three years prior to retirement and the number of years service with the employer. The longer the service, the higher the benefit.

Example 10.4 *Outline the main form of retirement benefit paid from a defined benefit fund.*

Solution 10.4 *A common benefit for a superannuation fund paying a defined benefit* *would be a pension of*

$$\frac{n}{60}FAS$$

where n is the number of years membership of the fund and FAS is the average salary over the three years prior to retirement. A member with 40 years service and salaries in the three years prior to retirement of 80,000, 85,000, and 75,000 would receive a pension of

$$\frac{40}{60} \times 80,000 = 53,333 \ p.a$$

from a fund with a benefit based on $\frac{n}{60}FAS$, usually paid fortnightly.
A member with 15 years service would receive

$$\frac{15}{60} \times 80,000 = 20,000 \ p.a.$$

In a defined benefit fund, the Trustees ensure that the superannuation fund will be able to pay the defined benefits by requiring the employer or sponsor to contribute sufficient money into the fund. The contribution is determined by the actuary to the fund. The amount contributed, along with the investment earnings on the assets of the fund, will have to be sufficient to pay the benefits under the Trust Deed. The benefits paid to members of the fund do not depend directly on the investment returns earned by the superannuation fund. The risk of low investment earnings is met by the employer who will normally make higher contributions into the fund in these circumstances. In theory the benefits may be reduced but this is not likely to be allowed by the Trustees.

A defined contribution (DC) fund specifies the contribution to the fund to be made by the employer and the employee as a percentage of the employee's salary. The amount of the benefit payable on retirement is not defined. It is determined by the amount that the contributions into the fund accumulate to, allowing for investment earnings, less the charges made for administration costs and the cost of death and disability cover. A defined contribution fund will usually provide life insurance cover as a multiple of salary or for a specified sum insured and the cost of this will be charged against the contributions made into the fund for an employee.

Governments usually provide favorable taxation treatment for superannuation savings through the second tier provision. In some countries the contributions made by an employer are fully tax deductible and not taxed as income for the employee. The investment earnings are usually not taxed and the final benefit is taxed when it is paid. This results in a deferral of taxation until retirement. Other countries have a less generous taxation treatment of superannuation funds. The taxation rules for these funds are complex.

Example 10.5 *How does the Australian government tax superannuation funds at present?*

Solution 10.5 *In Australia the government taxes contributions into superannuation funds at 15% as income of the fund, taxes individuals on contributions above a specified level as income, taxes the investment income of the funds at 15% and taxes the benefits when they are paid at a reduced rate, giving credit for the 15% tax already paid.*

Because there is a favorable taxation treatment, governments usually limit the amount of the retirement benefits that receive favorable taxation treatment.

Example 10.6 *Briefly outline the tax treatment for retirement benefits in Australia.*

Solution 10.6 *These limits are called Reasonable Benefit Levels or RBL's in Australia. In 1999 the limits were $485,692 if the benefit is taken as a lump sum and $971,392 if the benefit is taken as a pension. The pension receives more favorable treatment since the government is trying to encourage people to take pension benefits. Often the lump sum benefit is consumed quickly and the individual will then qualify for the age pension since the means test will no longer prevent them from receiving the pension.*

The third pillar of retirement provision is provided from the private savings of each individual in addition to the superannuation guarantee.

Benefits for retirement from these different levels of retirement provision can be financed on either a *funded* or a *pay-as-you go* basis. If funds are set aside to meet the future benefits from current income then the benefits are said to be funded. The fund will accumulate assets until benefits are paid.

If the cost of the benefits is just paid from future income as and when the benefits are due then the benefits are financed on a pay-as-you-go basis.

There is a current debate over whether or not retirement benefits should be funded or not. Governments in most countries do not fund the age pension or other social security benefits. These benefits are met from taxation revenue as and when the benefits have to be paid on a pay-as-you-go basis.

On the other hand, most employer superannuation funds are required to fund the benefits by setting aside contributions into the fund which, along with investment earnings, will be sufficient to meet the future expected benefits.

If future benefits are met on a pay-as-you-go basis then this will mean that the next generation will be paying the taxes to meet the aged pension and other government provided benefits for the current generation. This raises questions about *inter-generational equity* which reflects the relative cost of benefits provided for one generation and paid for by another.

10.3 Health Care and Disability

Health care is usually provided through both a public hospital and medical scheme and a private hospital and medical scheme in many countries. The private scheme provides a higher level of treatment and more choice. It is possible to purchase health insurance to cover private hospital and medical treatment.

Example 10.7 *Briefly explain how health care is financed in Australia.*

Solution 10.7 *In Australia, public health care for hospital and medical treatment is available and paid for partly by a Medicare levy based on taxable income. Private hospital treatment and additional health benefits are available at full cost to individuals without private insurance. Health insurance is available for private hospital and other health benefits.*

Community rating is used for the private health insurance in Australia. This is a system where everyone pays the same rate regardless of their risk. In order for these systems to work it is necessary for the system to be compulsory, otherwise the better risks will not join the system. The new private health insurance arrangements in Australia will be based on a *life time community rating* scheme. This will allow a different contribution rate to apply according to the age that an individual joins a health insurance fund. This will reflect the risk better than the current arrangements.

As the population ages there is an increasing need for a provision to be made for the costs of long term care after retirement. Insurance companies in some countries have introduced policies that provide income benefits for home and nursing home care. These policies pay benefits based on the number of *Activities of Daily Living (ADL's)* that the individual is unable to perform.

During an individual's working life, in the event of disability or sickness there will often be a substantial loss of income if this prevents them from working. Insurance companies sell *disability income insurance* to cover this risk. These policies are also called *permanent health insurance* (PHI) policies in some countries. The typical policy will pay up to 75% of an individual's income if they are disabled and unable to perform any occupation, or perhaps their own occupation. These benefits are usually paid to retirement age as long as the insured continues to be disabled.

10.4 Actuarial Modelling

The cost of providing retirement benefits from a superannuation fund must be assessed to determine if the benefits are affordable and to determine the contribution that should be set aside if the benefits are to be funded. This involves the determination of the expected present value of future retirement, ill-health, death and resignation benefits from a superannuation fund. The main factor that needs to be allowed for is that the benefits are usually linked to salary so that it is necessary to project future salaries of individual members of a fund in order to determine the benefit payments.

10.4.1 Salary Growth and Inflation

Defined benefits depend on future salaries. Disability income benefits also depend on future salaries. Taxation revenue will also depend on future salaries. In order to determine expected present values of benefits and revenues for superannuation and disability benefits that depend on salary we need to determine future salaries.

In order to determine expected future salaries it is necessary to project the future salary and determine the probability distribution of the future salary. Most companies have salary scales for different grades in the company and employees often progress through these grades by promotion and normal increments. In the early years of an individual's working career the salary progression can be forecast based on the salary scales. Later in a career the salary increases will depend on merit and performance and the salary progression will be less easy to forecast.

As well as promotional increases, individuals will receive increases due to community wide inflation increases.

In order to forecast future salaries, actuaries use *salary scales* that give the expected increase in salary by age. These scales are an index of relative salaries at each age based on company salary scales. Salary increases are usually higher for younger ages. Salary scales will vary from one superannuation fund to another. These salary scales also reflect expected productivity increases.

Assume that the current annual salary of a member aged x of a superannuation fund is S_x and that the salary scale is given by a table of values for each age equal to s_y for $y = 15$ to 65, where it is assumed that the youngest age in the fund is 15 and the oldest age is 65 (retirement age). The member's expected salary at age $x+t$, ignoring community wide salary inflation, will be

$$S_x \frac{s_{x+t}}{s_x}$$

Example 10.8 *A 20 year-old member of an employer superannuation fund has a current salary of \$35,000. The salary scale for ages 20 to 30 is as follows*

Age	***Salary Scale***
20	*1.25*
21	*1.31*
22	*1.39*
23	*1.48*
24	*1.61*
25	*1.74*
26	*1.82*
27	*1.91*
28	*1.97*
29	*2.01*
30	*2.03*

Determine the member's expected salary at ages 25 and 30.

Solution 10.8 *The member's expected salary at age 25 will be*

$$35,000\frac{1.74}{1.25} = 48,720$$

and at age 30 it will be

$$35,000\frac{2.03}{1.25} = 56,840$$

The salary scale is based on expected future salaries in a company.

As well as increase due to salary scales, individual's salaries will increase with community wide salary inflation. Assume that the community wide salary inflation rate is f p.a., then the future salary at age $x+t$ (a random variable) for a member of a superannuation fund currently aged x with current salary S_x will be

$$S_{x+t} = S_x\left(1+f\right)^t$$

The actual future salary will be a random variable and will vary from individual to individual. Future salaries can not be negative, and because they grow exponentially, a log-normal distribution is often used as a probability distribution for salary. The ratio of the salary at age $x+t$ to the salary at age x is

$$\frac{S_{x+t}}{S_x} = \frac{S_{x+1}}{S_x}\frac{S_{x+2}}{S_{x+1}}\cdots\frac{S_{x+t}}{S_{x+t+1}}$$

Taking logs of both sides we have

$$\begin{aligned} & \ln\left[\frac{S_{x+t}}{S_x}\right] \\ = & \ln\left[\frac{S_{x+1}}{S_x}\right] + \ln\left[\frac{S_{x+2}}{S_{x+1}}\right] + \cdots + \ln\left[\frac{S_{x+t}}{S_{x+t-1}}\right] \end{aligned}$$

Recall that $\ln\left[\frac{S_{x+1}}{S_x}\right]$ is the *continuous compounding salary growth rate* since if we let

$$\delta_f = \ln\left[\frac{S_{x+k}}{S_{x+k-1}}\right]$$

then

$$S_{x+k} = S_{x+k-1}e^{\delta_f}$$

and we see that δ_f is like a force of interest and is in fact the *force of inflation.*

We then have

$$\begin{aligned} \frac{S_{x+t}}{S_x} &= e^{\delta_f}e^{\delta_f}\cdots e^{\delta_f} \\ &= e^{t\times\delta_f} \end{aligned}$$

or

$$S_{x+t} = S_x e^{t\times\delta_f}$$

Example 10.9 *Assume that the continuous compounding salary growth rate is 3% p.a.. Project the future salary at age 65 for a person aged 20 on a salary of $35,000 assuming the same salary growth rate in each year.*

Solution 10.9 *The projected salary at age 65 would be*

$$\begin{aligned} & 35,000e^{45\times 0.03} \\ = & \; 35,000e^{1.35} \\ = & \; 35,000 \times 3.85743 \\ = & \; 135,010 \end{aligned}$$

Exercise 10.1 *Determine the projected future salary at age 65 for a 20 year old on a current salary of $35,000 assuming a continuous compounding salary growth rate of 4% p.a in each year.*

Exercise 10.2 *How any years will it take for the salary to double for a 20 year old on a current salary of $35,000.*

We can allow the continuous compounding salary growth rate to be different in each year, in which case we have

$$\begin{aligned} \frac{S_{x+t}}{S_x} &= e^{\delta_{1f}} e^{\delta_{2f}} \cdots e^{\delta_{tf}} \\ &= e^{\sum_{k=1}^{t} \delta_{kf}} \end{aligned}$$

or

$$S_{x+t} = S_x e^{\sum_{k=1}^{t} \delta_{kf}}$$

Assume that the ratio of salaries from age y to age $y+1$ has a $Log-normal\left(\mu_y, \sigma\right)$ distribution and that these salary ratios are all independent. This means that $\ln\left[\frac{S_{x+1}}{S_x}\right]$ has a normal distribution with mean μ_x and variance σ^2, $\ln\left[\frac{S_{x+2}}{S_{x+1}}\right]$ has a normal distribution with mean μ_{x+1} and variance σ^2, and so on.

The following result, which will be proved in later actuarial subjects, is very useful.

The sum of independent normal random variables has a normal distribution with mean equal to the sum of the expected values of the individual normal random variables and variance equal to the sum of the variances of the individual normal random variables.

Using this result we can see that $\ln\left[\frac{S_{x+t}}{S_x}\right]$ has a normal distribution with expected value $\sum_{y=x}^{y=x+t-1} \mu_y$ and variance $t\sigma^2$.

Thus $\left[\frac{S_{x+t}}{S_x}\right]$ has a $Log-normal\left(\sum_{y=x}^{y=x+t-1} \mu_y, \sqrt{t}\sigma\right)$ distribution.

We then have

$$E\left[\frac{S_{x+t}}{S_x}\right] = \exp\left[\sum_{y=x}^{y=x+t-1} \mu_y + \frac{1}{2}t\sigma^2\right]$$

or, since at age x the value of S_x is known (and is not random), we have

$$E\left[S_{x+t}\right] = S_x \exp\left[\sum_{y=x}^{y=x+t-1} \mu_y + \frac{1}{2}t\sigma^2\right]$$

Example 10.10 *Assume that the continuous compounding salary growth rate for any year has a normal distribution with expected value 0.03 and standard deviation 0.01 and each years's growth rate is independent. Determine the expected salary at age 65 for a 20 year old individual with a current salary of $35,000.*

Solution 10.10 *The expected salary at age 65 will be*

$$\begin{aligned} E\left[S_{65}\right] &= 35,000 \exp\left[45 \times 0.03 + \frac{1}{2}45\left(0.01\right)^2\right] \\ &= 35,000e^{1.35225} \\ &= 35,000 \times 3.86611 \\ &= 135,314 \end{aligned}$$

Exercise 10.3 *Determine the expected future salary at age 65 for a 20 year old on a current salary of $35,000 assuming the continuous compounding salary growth rate for any year is independent and has a normal distribution with expected value 0.04 and standard deviation 0.01.*

Assuming a log-normal distribution for the salary growth rate, the variance of $\ln\left[\frac{S_{x+t}}{S_x}\right]$ increases linearly with t. For large values of t this can mean that the variance becomes too large. It is possible to have the variability increase at a slower rate and eventually become constant by allowing the salary growth rate to depend on previous salary growth rates. For example, a lower expected growth rate might be assumed for year $t+1$ whenever the actual salary growth rate is above that expected in year t and vice versa. Distributions with such dependence from one year to the next are studied in later actuarial subjects.

10.4.2 The Service Table - Survival and Other Probabilities

The value of future benefits of superannuation and disability funds depend on the chance that an individual will reach retirement, die, become sick or withdraw from the fund. To determine the expected present value of benefits we require the probabilities that these events occur.

For superannuation plans, actuaries use an extension of the life table to include the probabilities of death and disability as well as the probabilities that a member will withdraw from the fund at each age. The table that shows these probabilities by age is called a *service table*.

The following table is an example of a table of these probabilities. It shows the probability that a member of the fund will leave the fund at any age from either withdrawal (resignation) or from death and disability. The sum of these two probabilities is the probability that they leave the fund for each age. The probability that a member will receive a retirement benefit is the probability that they do not leave the fund by resignation or death and disability before retirement age (65).

Age	Resignation	Death and Disability
20	0.16650	0.00090
21	0.16020	0.00089
22	0.15390	0.00083
23	0.14760	0.00075
24	0.14130	0.00067
25	0.13500	0.00063
26	0.12870	0.00063
27	0.12240	0.00064
28	0.11610	0.00066
29	0.10980	0.00068
30	0.10350	0.00072
31	0.09720	0.00077
32	0.09090	0.00082
33	0.08460	0.00087
34	0.07830	0.00093
35	0.07200	0.00101
36	0.06795	0.00110
37	0.06390	0.00121
38	0.05985	0.00132
39	0.05580	0.00146
40	0.05175	0.00163
41	0.04770	0.00183
42	0.04365	0.00205
43	0.03960	0.00231
44	0.03555	0.00262
45	0.03150	0.00297
46	0.02745	0.00338
47	0.02340	0.00385
48	0.01935	0.00439
49	0.01530	0.00501
50	0.01125	0.00572
51	0.00900	0.00655
52	0.00675	0.00751
53	0.00450	0.00860
54	0.00225	0.00983
55	0.00000	0.01123
56	0.00000	0.01282
57	0.00000	0.01462
58	0.00000	0.01665
59	0.00000	0.01895
60	0.00000	0.02154
61	0.00000	0.02443
62	0.00000	0.02784
63	0.00000	0.03185
64	0.00000	0.03659
65	0.00000	0.04215

The service table shows the probabilities that a life aged x will resign from a superannuation fund or die or become disabled for ages 20 to 65 and will show the probability that a member will remain in the fund for each age. These probabilities can be used to determine the other functions shown in a life table.

Example 10.11 *Using the service table given, determine the probability that a life aged 25 will resign in the following year and the probability that a life aged 55 will die or become disabled in the following year.*

Solution 10.11 *The probability that a life aged 25 will resign is 0.135 and the probability that a life aged 55 will die or become disabled will be 0.01123.*

Example 10.12 *Using the service table given, determine the probability that a member aged 40 will still be a member at age 45.*

Solution 10.12 *The probability that a member aged 40 will still be a member at age 41 will be* $1 - 0.05175 - 0.00163 = 0.94662$.
The probability that a member aged 41 will still be a member at age 42 will be $1 - 0.04770 - 0.00183 = 0.95047$.
Similarly, for age 42 the probability that the member will still be a member at age 43 is 0.95430 *and for age 43 the probability that they will be a member at the end of another year will be* 0.95809*. At age 44 the probability will be* 0.96183.
The required probability is

$$\begin{aligned} & 0.94662 \times 0.95047 \times 0.95430 \times 0.95809 \times 0.96183 \\ = \; & 0.79123 \end{aligned}$$

Example 10.13 *Determine the probability that a 60 year old will reach age 65 and receive a retirement benefit using the service table given.*

Solution 10.13 *The required probability is*

$$\begin{aligned} & (1 - 0.02154)(1 - 0.02443)(1 - 0.02784)(1 - 0.03185)(1 - 0.03659) \\ = \; & 0.86555 \end{aligned}$$

Exercise 10.4 *Assuming that 10000 members join a fund at age 20, use the table of probabilities given to construct a service table showing the expected number of members of the fund at each age from 21 through 65 and the expected number of years of membership of the fund for a new member entering the fund at each of these ages. (You will need to use a spreadsheet).*

Exercise 10.5 *Use the service table probabilities given to determine the probability that a new member of the fund entering at age 35 will leave the fund because of death at age 40.*

10.4.3 Defined benefits

Expected present value of benefits

The expected present value of the benefits for a defined benefit superannuation fund are determined using the benefit rules for the defined benefit payable on death and disability, withdrawal and retirement. The salary at each age of leaving the fund is forecast using a probability distribution, such as the log-normal distribution. This is used to determine the distribution of the benefits. The present value of these benefits is the projected value discounted for interest to the current date. The expected value of the future benefits can then be calculated using the service table probabilities to determine the probability that the benefits will be paid at each age.

As a simple example, consider a member aged x of a fund on a current salary of S_x that pays a lump sum retirement benefit of

$$k\frac{n}{60}S_{65}$$

where n is the number of years membership of the fund, k is a benefit multiple and S_{65} is the final salary at retirement.

Assume that the continuous compounding salary growth rate in year $x+t$, denoted by δ_{x+t}, has a $Normal\left(\mu,\sigma\right)$ distribution and these growth rates are independent. Note that this means that the salary at each future age has a $Log-normal$ distribution.

The probability that a member of the fund aged x will remain a member to age $x+t$ is denoted ${}_t\left(ap\right)_x$.

The interest rate for determining the present value is assumed to be i p.a. The interest rate is assumed to be fixed and known.

The **expected present value of the retirement benefit at age 65** for a member aged x on a current salary S_x will be

$$\begin{aligned}
& E\left[\left(\frac{1}{1+i}\right)^{(65-x)}k\frac{(65-x)}{60}S_{65}\right] \\
= \; & E\left[\left(\frac{1}{1+i}\right)^{(65-x)}k\frac{(65-x)}{60}S_{65}| \begin{array}{l}\text{life survives}\\ \text{to age 65}\end{array}\right]\Pr\left[\begin{array}{l}\text{life survives}\\ \text{to age 65}\end{array}\right] \\
= \; & \left(\frac{1}{1+i}\right)^{(65-x)}k\frac{(65-x)}{60}E\left[S_{65}\right]_{65-x}\left(ap\right)_x
\end{aligned}$$

This is so since the benefit that will be paid at age 65, provided the member survives to age 65, will be based on $(65-x)$ years of service and the benefit will be

$$k\frac{(65-x)}{60}S_{65}$$

The present value of this benefit will be obtained by discounting the benefit from age 65 to the current age x at the interest rate i. The present value factor is

$$\left(\frac{1}{1+i}\right)^{(65-x)} = v^{65-x}$$

where

$$v = \frac{1}{1+i}$$

The expected value has to allow for the probability that the member receives the retirement benefit and the expected value of the future salary. The values of

$$\left(\frac{1}{1+i}\right)^{(65-x)} \frac{(65-x)}{60}$$

are known and are not random variables (given that we know the current age x and that the interest rate is fixed and known) and can be taken outside of the expected value operator. The retirement benefit at age 65 is zero for every other age.

To determine the expected salary at age 65 we know that by definition

$$S_{65} = S_x e^{\sum_{t=x}^{64} \delta_t}$$

Therefore

$$\ln\left(\frac{S_{65}}{S_x}\right) = \sum_{t=x}^{64} \delta_t$$

Since $\sum_{t=x}^{64} \delta_t$ is the sum of $(65-x)$ independent normally distributed random variables each with expected value μ and variance σ^2 we have that $\sum_{t=x}^{64} \delta_t$ is normally distributed with expected value $(65-x)\,\mu$ and variance $(65-x)\,\sigma^2$. Thus $\left(\frac{S_{65}}{S_x}\right)$ has a *Lognormal* $\left((65-x)\,\mu, \sqrt{(65-x)}\sigma\right)$ distribution. Therefore

$$\begin{aligned} E\left(\frac{S_{65}}{S_x}\right) &= \exp\left[(65-x)\,\mu + \frac{1}{2}(65-x)\,\sigma^2\right] \\ &= \exp\left((65-x)\left[\mu + \frac{1}{2}\sigma^2\right]\right) \\ &= \left(\exp\left[\mu + \frac{1}{2}\sigma^2\right]\right)^{(65-x)} \end{aligned}$$

and

$$E\left[S_{65}\right] = S_x \left(\exp\left[\mu + \frac{1}{2}\sigma^2\right]\right)^{(65-x)}$$

Thus the expected present value of the retirement benefit will be

$$\frac{\left(\exp\left[\mu+\frac{1}{2}\sigma^2\right]\right)^{(65-x)}}{(1+i)^{(65-x)}}k\frac{(65-x)}{60}S_{x\ 65-x}(ap)_x$$

If we let

$$\exp\left[\mu+\frac{1}{2}\sigma^2\right]=1+f$$

then the expected present value of the retirement benefit will be

$$\begin{aligned} & {}_{65-x}(ap)_x\frac{(1+f)^{(65-x)}}{(1+i)^{(65-x)}}k\frac{(65-x)}{60}S_x \\ = & {}_{65-x}(ap)_x\left[\frac{1+f}{1+i}\right]^{(65-x)}k\frac{(65-x)}{60}S_x \end{aligned}$$

Example 10.14 *A superannuation fund pays a lump sum retirement benefit at age 65 of*

$$8\times\frac{service}{60}$$

Determine the expected present value of the retirement benefit for a 40 year old member on a current salary of \$55,000 with 10 years current service in the fund. Assume that the probability of remaining a member of the fund to age 65 for a 40 year old is 0.55 and that the continuous compounding salary growth rate in each year is independent and has a $Normal\,(0.04,0.01)$ *distribution. The current interest rate for present valuing the retirement benefit is assumed to be 7% p.a.*

Solution 10.14 *We have*

$${}_{65-x}(ap)_x=0.55$$

$$\begin{aligned} 1+f &= \exp\left[\mu+\frac{1}{2}\sigma^2\right] \\ &= \exp\left[0.04+\frac{1}{2}(0.01)^2\right] \\ &= 1.04086 \end{aligned}$$

$$\begin{aligned} \left[\frac{1+f}{1+i}\right]^{(65-x)} &= \left(\frac{1.04086}{1.07}\right)^{25} \\ &= 0.972769^{25} \\ &= 0.501468 \end{aligned}$$

$$
\begin{aligned}
k\frac{(65-x)}{60}S_x &= 8 \times \frac{35}{60} \times 55,000 \\
&= 4.66667 \times 55,000 \\
&= 256,667
\end{aligned}
$$

The expected present value of the benefit will be

$$0.55 \times 0.501468 \times 256,667 = 70,791$$

Exercise 10.6 *A superannuation fund pays a lump sum retirement benefit at age 65 of*

$$8 \times \frac{service}{60}$$

Determine the expected present value of the retirement benefit for a 50 year old member on a current salary of \$65,000 with 15 years current service in the fund. Assume that the probability of remaining a member of the fund to age 65 for a 50 year old is 0.65 and that the continuous compounding salary growth rate in each year is independent and has a $Normal\,(0.03, 0.01)$ *distribution. The current interest rate for present valuing the retirement benefit is assumed to be 7% p.a.*

Similar methods are used to value the death and disability benefits and any resignation benefits that are a multiple of salary and related to service. These methods for determining the expected present value of future superannuation fund benefits are covered in detail in later actuarial subjects. Once all the benefits have been valued it is necessary to determine how the costs of the benefits will be met.

Actuaries usually separate the value of *accrued benefit obligations (ABO's)* from the expected present value of benefits accruing from future service. The expected present value of the benefits paid from the fund in the future will include a portion that is in respect of the past service or membership of the fund as at the current date and a portion that relates to future service or membership after the current date. The portion that relates to past service is the ABO or past service obligation.

Expected present value of contributions

The benefits would normally be funded by contributing a percentage of salary at the end of each year provided that the individual is still a member of the fund. The expected present value at age x of a contribution of 1% of the salary at age $x+t$ will be

$$
\begin{aligned}
& 0.01\frac{{}_t\,(ap)_x\, S_x \left(\exp\left[\mu + \frac{1}{2}\sigma^2\right]\right)^t}{(1+i)^t} \\
= \;& (0.01 S_x)_t\,(ap)_x \left[\frac{1+f}{1+i}\right]^t
\end{aligned}
$$

This is so since the probability that a member currently aged x will be in the fund at age $x+t$ will be ${}_t(ap)_x$. The expected value of the salary at age $x+t$ will be $S_x\left(\exp\left[\mu+\frac{1}{2}\sigma^2\right]\right)^t$. The expected present value of 1% of salary paid at age $x+t$ will therefore be the product of the probability of payment times the expected payment and then present valued to the current date.

The expected present value of a contribution of 1% of salary for a member aged x as long as they are in the fund will be

$$(0.01S_x)\sum_{t=1}^{65-x} {}_t(ap)_x\left[\frac{1+f}{1+i}\right]^t$$

since we just need to sum over all possible future ages that the member will be in the fund.

If we let

$$\frac{1}{1+j}=\frac{1+f}{1+i}$$

then

$$j=\frac{1+i}{1+f}-1$$

and the expected present value of a contribution of 1% of salary for a member aged x as long as they are in the fund is

$$(0.01S_x)\sum_{t=1}^{65-x}\frac{{}_t(ap)_x}{(1+j)^t}$$

Note that this is very similar to the expected present value of a term life annuity except that here the payments depend on the future salary and are not level payments. The future salary is also a random variable.

Exercise 10.7 *Use the service table probabilities given earlier to determine the expected present value of 1% of salary, as a percentage of the member's current salary, payable while the member remains in the fund for each age of entry. Assume the continuous compounding salary growth rate in each year is independent and has a* $Normal\,(0.03, 0.01)$ *distribution. The current interest rate for present valuing the retirement benefit is assumed to be 7% p.a.*

10.4.4 Accumulation funds

In the case of accumulation funds, the projection of the assets available for retirement allowing for investment returns and future contributions is required in order to assess the amount of retirement income that the fund will provide. This involves projecting the accumulation of the fund to the retirement age. The accumulated funds are often determined as a multiple of projected final salary to indicate the relative amount of the assets available for retirement compared with the salary received prior to retirement.

The other important issue in these funds is determining the asset allocation that will maximize the retirement benefit allowing for the risk aversion of the individual. Some individuals will prefer to have their retirement funds heavily invested in the sharemarket whereas others will prefer to have their funds invested mainly in fixed interest investments. Members of a fund who are closer to retirement age often prefer a more conservative investment strategy for their retirement funds. Younger members often prefer a higher proportion of sharemarket investments.

Measured over long periods of time, the expected returns on sharemarket investments are higher than those for more conservative investments such as fixed interest investments and cash. However the returns on sharemarket investments are much more variable than for the more conservative investments and the probability of a negative return (where you actually lose wealth) is much higher with sharemarket investments.

The determination of the best investment strategy for an individual to adopt for their retirement provision can be derived by maximizing the expected utility of future wealth. The proportions invested in the different asset classes are varied in order to maximize the expected utility of wealth on the retirement date.

10.4.5 The Ageing Population

In order to determine the value of social security and aged care health benefits it is necessary to project the future population of a country by individual age groups. These projections need to allow for births (fertility), deaths (mortality) as well as migration. The assumptions used in the projection need to be selected with some care.

In many countries, mortality levels have been improving over recent years for a number of reasons including better health care. There is some question whether or not this mortality improvement will continue into the future.

Fertility levels in most developed countries have been declining over recent years. Females have been deferring having children and having fewer children than in the past.

Methods of projecting the future population have been developed and studied in detail in demography.

10.5 Conclusions

This chapter has briefly outlined the types of retirement and health care benefits provided and the main actuarial techniques used to assess the value of these benefits. The costs of retirement income provisions and aged care are increasing as the population ages and the assessment of the costs and the financing of these costs is a major area requiring actuarial involvement in the future.

10.6 Solutions to Exercises

Ex 10.1 *Projected salary at age 65*

$$\begin{aligned} & 35,000e^{45\times 0.04} \\ = \; & 35,000e^{1.8} \\ = \; & 35,000 \times 6.049647 \\ = \; & 211,738 \end{aligned}$$

Ex 10.2 *Salary will double in t years if the continuous compounding salary growth rate is δ when*

$$35,000e^{t\times\delta} = 70000$$

or

$$e^{t\times\delta} = 2$$

or

$$\begin{aligned} t &= \frac{\ln 2}{\delta} \\ &= \frac{0.69315}{\delta} \end{aligned}$$

If we use $\delta = 0.04$ then $t = 17.33$ years. The following table shows how long it takes for salary to double for different salary growth rates.

Salary Growth Rate (Continuous)	Equivalent Annual Effective Growth Rate	Time to double salary (years)
1.00%	1.01%	69.31
2.00%	2.02%	34.66
3.00%	3.05%	23.10
4.00%	4.08%	17.33
5.00%	5.13%	13.86
6.00%	6.18%	11.55
7.00%	7.25%	9.90
8.00%	8.33%	8.66
9.00%	9.42%	7.70
10.00%	10.52%	6.93

Ex 10.3 *Expected future salary at age 65 for a 20 year old on \$35,000*

$$\begin{aligned} & 35,000 \exp\left(45 \times 0.04 + 45 \times \frac{1}{2} \times 0.01^2\right) \\ = \; & 35,000 \exp(1.80225) \\ = \; & 35,000 \times 6.06327 \\ = \; & 212,215 \end{aligned}$$

Ex 10.4 *The following table shows the expected number who leave the fund due to withdrawal (rx), the expected number who leave due to death or disability (dx), the expected number in the fund (lx), the expected number between age x and x+1 (Lx), the number of future years membership of the lx members aged x (Tx) and the expected number of years membership ax=Tx/lx.*

Age	rx	dx	lx	Lx	Tx	ax
20	1665.00	9.00	10000.00	9163.00	80672.81	8.07
21	1333.83	7.41	8326.00	7655.38	71509.81	8.59
22	1074.96	5.80	6984.76	6444.39	63854.43	9.14
23	871.43	4.43	5904.01	5466.08	57410.04	9.72
24	710.48	3.37	5028.15	4671.23	51943.96	10.33
25	582.43	2.72	4314.31	4021.73	47272.73	10.96
26	479.94	2.35	3729.16	3488.01	43251.00	11.60
27	397.42	2.08	3246.86	3047.12	39762.99	12.25
28	330.58	1.88	2847.37	2681.14	36715.87	12.89
29	276.14	1.71	2514.91	2375.99	34034.73	13.53
30	231.54	1.61	2237.06	2120.49	31658.74	14.15
31	194.78	1.54	2003.92	1905.76	29538.25	14.74
32	164.31	1.48	1807.59	1724.70	27632.50	15.29
33	138.90	1.43	1641.80	1571.64	25907.80	15.78
34	117.57	1.40	1501.48	1442.00	24336.16	16.21
35	99.54	1.40	1382.51	1332.05	22894.17	16.56
36	87.08	1.41	1281.58	1237.33	21562.12	16.82
37	76.24	1.44	1193.08	1154.24	20324.79	17.04
38	66.76	1.47	1115.40	1081.29	19170.55	17.19
39	58.43	1.53	1047.17	1017.19	18089.26	17.27
40	51.09	1.61	987.21	960.86	17072.07	17.29
41	44.58	1.71	934.51	911.37	16111.21	17.24
42	38.77	1.82	888.23	867.93	15199.83	17.11
43	33.57	1.96	847.64	829.87	14331.90	16.91
44	28.87	2.13	812.11	796.61	13502.03	16.63
45	24.61	2.32	781.11	767.65	12705.42	16.27
46	20.70	2.55	754.19	742.56	11937.76	15.83
47	17.10	2.81	730.94	720.98	11195.20	15.32
48	13.76	3.12	711.02	702.58	10474.22	14.73
49	10.62	3.48	694.14	687.09	9771.65	14.08
50	7.65	3.89	680.04	674.27	9084.56	13.36
51	6.02	4.38	668.50	663.30	8410.28	12.58
52	4.44	4.94	658.11	653.41	7746.98	11.77
53	2.92	5.58	648.72	644.47	7093.57	10.93
54	1.44	6.29	640.22	636.36	6449.10	10.07
55	0.00	7.10	632.49	628.94	5812.74	9.19
56	0.00	8.02	625.39	621.38	5183.80	8.29
57	0.00	9.03	617.37	612.86	4562.43	7.39
58	0.00	10.13	608.34	603.28	3949.57	6.49
59	0.00	11.34	598.21	592.55	3346.29	5.59
60	0.00	12.64	586.88	580.56	2753.75	4.69
61	0.00	14.03	574.24	567.22	2173.19	3.78
62	0.00	15.60	560.21	552.41	1605.97	2.87
63	0.00	17.35	544.61	535.94	1053.56	1.93
64	0.00	19.29	527.27	517.62	517.62	0.98
65			507.97	0.00	0.00	

Ex 10.5 *Probability that a new member entering the fund at age 35 will leave the fund because of death at age 40 is*

$$\begin{aligned} \frac{d_{40}}{l_{35}} &= \frac{1.61}{1382.51} \\ &= 0.001164 \end{aligned}$$

Ex 10.6 *The expected future salary is*

$$\begin{aligned} & 65,000 \exp\left(15 \times 0.03 + \frac{1}{2} \times 15 \times 0.01^2\right) \\ = \; & 102,016.8 \end{aligned}$$

The expected future benefit is

$$\begin{aligned} & 8 \times \frac{30}{60} \times 102,016.8 \\ = \; & 408,067 \end{aligned}$$

The present value of the expected future benefit is

$$\frac{408,067}{(1.07)^{15}} = 147,902$$

and the expected present value of the retirement benefit becomes

$$0.65 \times 147,902 = 96,136$$

Ex 10.7 *We can work out an effective interest rate allowing for the expected salary growth rate as follows*

$$1 + f = e^{0.03 + \frac{1}{2} 0.01^2} = 1.030506$$

so that

$$\begin{aligned} \frac{1}{1+j} &= \frac{1.030506}{1.07} \\ &= 0.96309 \end{aligned}$$

Note we assume that salary contributions are paid at the end of the year. The expected present value of 1% of salary while the member remains in the fund as a per cent of current salary at each age is

$$\sum_{t=1}^{65-x} \frac{{}_t(ap)_x}{(1+j)^t}$$

Note that we can calculate this recursively. The table below shows the values for this expression for each age

Age	**Expected Present Value of 1% of Salary (as a percent of Salary)**
20	5.19
21	5.47
22	5.77
23	6.09
24	6.42
25	6.77
26	7.13
27	7.51
28	7.89
29	8.27
30	8.66
31	9.04
32	9.40
33	9.75
34	10.07
35	10.35
36	10.59
37	10.82
38	11.01
39	11.18
40	11.31
41	11.41
42	11.46
43	11.47
44	11.44
45	11.34
46	11.20
47	11.00
48	10.74
49	10.42
50	10.05
51	9.61
52	9.14
53	8.63
54	8.08
55	7.49
56	6.86
57	6.22
58	5.55
59	4.86
60	4.15
61	3.40
62	2.62
63	1.80
64	0.93

Chapter 11
REGULATION AND PROFESSIONAL ETHICS

11.1 Learning Objectives

The main objectives of this chapter are:

- to introduce the professional code of conduct, and
- to raise issues of professional ethics.

11.2 Professional Code of Conduct

The professional actuarial bodies around the world have codes of conduct. The Institute of Actuaries of Australia has a code of conduct as well. Members of the actuarial profession are expected to place the public interest above other considerations in their work. They are also expected not to carry out any work that they are not qualified to do. They should not be involved in giving advice if they have a conflict of interest.

11.2.1 Professional standards and guidance notes

The actuarial profession has defined acceptable approaches to certain tasks carried out by actuaries in professional standards and guidance notes. Professional standards cover matters including:

- investment performance measurement and presentation,
- investment advice,
- determination of life insurance policy liabilities,
- determination of minimum surrender values
- actuarial reports and advice on outstanding claims in general insurance, and
- investigation of defined benefit superannuation funds.

The professional standards set out the matters that should be addressed by an actuary in specified actuarial work.

Guidance notes provide assistance to actuaries in tasks often required by legislation.

11.2.2 Disciplinary action

The professional bodies have a disciplinary procedure to be followed when a member of the professional body is accused of professional misconduct.

11.3 Professional Ethics

The value of a professional qualification reflects the reputation that the members of the profession have established over many years. The actuarial profession has built a highly regarded reputation in the financial services industry for its unbiased and professional approach.

An individual's reputation is one of their most valuable assets. They should always act so as not to lose their reputation and to not put at risk the reputation of the profession.

Professional ethics and appropriate professional conduct will often be learned by experience. There are many ways that an actuary can ensure that their behaviour is ethical. If in doubt they should always discuss these issues with other members of the profession and if necessary should seek advice from senior members of the profession.

The public interest should always dominate the considerations of any professional including the actuary.

Actuaries are often involved in determining the equitable treatment of different parties to a financial agreement. They should always be unbiased in their actions and recommendations.

An actuary is a member of a profession. They are expected to maintain the highest standards of professional behaviour. This is governed by professional code of conduct as well as professional standards and guidance notes. Actuaries will only maintain their own reputations and enhance the reputation of the profession if they observe the highest standards of ethical behaviour.

11.4 Equity Funding

The Equity Funding case was one of the largest insurance frauds that has ever occurred. It is now popularised in the "Billion Dollar Bubble" film starring James Wood. An actuary was sent to jail over his involvement in the illegal activities of the company.

Equity Funding was part of a conglomerate that had interests in different areas and was actively involved in takeovers and growth. The Equity Funding company marketed the early unbundled life insurance contracts in North America. Policyholders were given a loan, a small part of which was used to purchase term insurance, and the remainder was used to invest in the sharemarket (equity market). Policyholders were not required to invest any of their own funds in the product. They were told that the value of the sharemarket investment in the future would be more than sufficient to repay the loan.

The conglomerate was taking over companies that were contributing losses rather than profits to the conglomerate's bottom line. This meant that the conglomerate was facing cash flow problems. It could hide the losses with clever accounting but could not hide the need for cash to finance the operation of the companies.

Equity Funding was able to generate cash flow by selling insurance policies to reinsurance companies. These reinsurance companies provide up-front financing for insurance companies by paying them commissions and contributing towards meeting the initial costs of the insurance company. The sales of insurance business involves high initial expenses that must be financed. This is called *new business strain*.

Equity Funding started creating and reinsuring false policies in order to generate cash to finance the losses and cash flow problems of the other businesses. The original intention was to buy back the policies when cash flow improved. But cash flow did not improve.

Once an insurance policy has been reinsured the insurance company will have to pay future premiums to the reinsurance company. They also will have to repay any financing of new business strain in future years. Because many of the policies that were being reinsured were not paying any premiums to Equity Funding, they eventually had to create false policies to sell to the reinsurance companies in order to pay future premiums on the previously reinsured business.

It was projected that, based on the sales of new (and false) policies by Equity Funding, they would have insurance on the whole of the US population within not too many years.

Eventually, an employee of the company informed the Regulators and Equity Funding was shut down. Many of the executives were sent to jail, including the actuary.

The Equity Funding executives were dominated by a charismatic Chief Executive of the conglomerate. It was very actively pursuing rapid growth by takeover. These characteristics are often found in corporate collapses and scandals.

11.5 Conclusions

Actuaries are members of a profession with a long history and a high reputation for honest and fair professional conduct. All members of the profession need to maintain the highest level of integrity in their professional work, observe the code of conduct and conduct themselves in an ethical manner.

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